Física Matemática I - (diurno) - FMA204

Exame 1: 16/10/17

GABARITO

1. (a)

$$\alpha_0 = \frac{L^2}{6}.$$

(b) Para todo $k > 0$,

$$\alpha_k = \frac{L^2}{\pi^2}\frac{(-1)^k}{k^2}.$$

(c) Para todo $k > 0$,

$$\beta_k = 0.$$

(d)

$$f(x) = \frac{L^2}{12} + \frac{L^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(2\pi\frac{kx}{L}\right).$$

(e) A série é convergente ponto-a-ponto, bem como absoluta e uniformemente convergente.

2. (a) A função é contínua em todo o intervalo, e é diferenciável em todos os postos exceto em $x = 0$ e $x = \pm L/2$.

$$\alpha_0 = \frac{4}{\pi}.$$

(b) Com $k = 2j$ e $j \in \{0, 1, 2, \ldots, \infty\}$,

$$\alpha_k = \frac{4}{\pi}\frac{1}{1-k^2}.$$

(c) Para todo $k > 0$,

$$\beta_k = 0.$$

(d) Com $k = 2j$ e $j \in \{0, 1, 2, \ldots, \infty\}$,

$$\begin{aligned} f(x) &= \frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{1-k^2}\cos\left(\frac{2\pi kx}{L}\right) \\ &= \frac{2}{\pi} + \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{1-4j^2}\cos\left(\frac{4\pi jx}{L}\right). \end{aligned}$$

(e) A série é convergente ponto-a-ponto, bem como absoluta e uniformemente convergente.

3. (a)

$$\widetilde{F}(\omega) = \frac{I_0}{\sqrt{2\pi}}.$$

(b)

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\ \mathrm{e}^{\imath \omega t}\, \widetilde{x}(\omega).$$

(c)

$$\left[-m\omega^2 + \imath\gamma\omega + k\right] \widetilde{x}(\omega) = \frac{I_0}{\sqrt{2\pi}}.$$

(d) Com $\omega_0^2 = k/m$,

$$\begin{aligned} \widetilde{x}(\omega) &= \frac{-I_0}{m\sqrt{2\pi}} \frac{1}{\omega^2 - \imath(\gamma/m)\omega - \omega_0^2} \\ &= \frac{-I_0}{m\sqrt{2\pi}} \frac{1}{(\omega-\omega_+)(\omega-\omega_-)}, \\ \omega_\pm &= \pm\sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2} + \imath\left(\frac{\gamma}{2m}\right). \end{aligned}$$

(e)

$$x(t) = \frac{I_0}{m} \frac{\sin\left[\sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}\, t\right]}{\sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}} \exp\left[-\left(\frac{\gamma}{2m}\right) t\right].$$

4. (a)

$$\alpha_0 = \frac{2L^2}{3}.$$

(b) Para todo $k > 0$,

$$\alpha_k = \frac{L^2}{\pi^2 k^2}.$$

(c) Para todo $k > 0$,

$$\beta_k = -\frac{L^2}{\pi k}.$$

(d)

$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(2\pi\frac{kx}{L}\right) - \frac{L^2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k} \sin\left(2\pi\frac{kx}{L}\right).$$

(e) O fator de $1/k^2$ na sub-série de cossenos é suficiente para que esta sub-série seja absoluta e uniformemente convergente. Por outro lado, o fator de $1/k$ na sub-série de senos é suficiente para que ela seja convergente ponto-a-ponto, mas não para que seja absoluta ou uniformemente convergente. Desta forma, a série como um todo é convergente ponto-a-ponto, mas não absoluta nem uniformemente convergente.

5. (a)

$$\begin{aligned} \widetilde{F}(\omega) &= \sqrt{\frac{\pi}{2}}\, F_0 \left[\delta(\omega-\omega_F)+\delta(\omega+\omega_F)\right], \\ x(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{\imath\omega t}\, \widetilde{x}(\omega). \end{aligned}$$

(b) Com $\omega_0^2 = k/m$,

$$\begin{aligned} \widetilde{x}(\omega) &= \sqrt{\frac{\pi}{2}}\, F_0\, \frac{\delta(\omega-\omega_F)+\delta(\omega+\omega_F)}{-m\omega^2+\imath\gamma\omega+k}, \\ x(t) &= \frac{F_0}{m} \frac{\cos(\omega_F t-\phi_0)}{\sqrt{\left(\omega_0^2-\omega_F^2\right)^2+(\gamma/m)^2\omega_F^2}}, \\ \cos(\phi_0) &= \frac{m\left(\omega_0^2-\omega_F^2\right)}{\sqrt{\left[m\left(\omega_0^2-\omega_F^2\right)\right]^2+\gamma^2\omega_F^2}}, \\ \sin(\phi_0) &= \frac{\gamma\omega_F}{\sqrt{\left[m\left(\omega_0^2-\omega_F^2\right)\right]^2+\gamma^2\omega_F^2}}. \end{aligned}$$

(c) Com $\omega_0 > \gamma/(2m)$,

$$x(t) = \left\{ A\cos\left[\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}\, t\right] + B\sin\left[\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}\, t\right] \right\} \exp\left[-\left(\frac{\gamma}{2m}\right) t\right].$$

(d)

$$\begin{aligned} x(t) = {} & \frac{F_0}{m\sqrt{\left(\omega_0^2-\omega_F^2\right)^2+(\gamma/m)^2\omega_F^2}} \times \\ & \times \left\{ \cos(\omega_F t-\phi_0) - \cos(\phi_0)\cos\left[\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}\, t\right] \right. \\ & \left. - \frac{[\gamma/(2m)]\cos(\phi_0)+\omega_F\sin(\phi_0)}{\sqrt{\omega_0^2-[\gamma/(2m)]^2}} \sin\left[\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}\, t\right] \right\} \exp\left[-\left(\frac{\gamma}{2m}\right) t\right]. \end{aligned}$$

(e) Com $z = \tau + \imath\sigma$ e $w(z) = x(\tau,\sigma) + \imath y(\tau,\sigma)$,

$$\begin{aligned} w(z) &= \frac{F_0}{m} \frac{\cos(\omega_F z-\phi_0)}{\sqrt{\left(\omega_0^2-\omega_F^2\right)^2+(\gamma/m)^2\omega_F^2}}, \\ x(\tau,\sigma) &= \frac{F_0}{m} \frac{\cos(\omega_F\tau-\phi_0)\cosh(\omega_F\sigma)}{\sqrt{\left(\omega_0^2-\omega_F^2\right)^2+(\gamma/m)^2\omega_F^2}}, \\ y(\tau,\sigma) &= -\frac{F_0}{m} \frac{\sin(\omega_F\tau-\phi_0)\sinh(\omega_F\sigma)}{\sqrt{\left(\omega_0^2-\omega_F^2\right)^2+(\gamma/m)^2\omega_F^2}}. \end{aligned}$$