# On the Sums of Inverse Even Powers of Zeros of Regular Bessel Functions

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**Abstract**

We provide a new, simple general proof of the formulas giving the infinite sums $\sigma(p,\nu)$ of the inverse even powers $2p$ of the zeros $\xi_{\nu k}$ of the regular Bessel functions $J_\nu(\xi)$, as functions of $\nu$. We also give and prove a general formula for certain linear combinations of these sums, which can be used to derive the formulas for $\sigma(p,\nu)$ by purely linear-algebraic means, in principle for arbitrarily large powers. We prove that these sums are always given by a ratio of two polynomials on $\nu$, with integer coefficients. We complete the set of known formulas for the smaller values of $p$, extend it to $p=9$, and point out a connection with the Riemann zeta function, which allows us to calculate some of its values.

## 1 Introduction

In boundary value problems involving the diffusion equation the following infinite sums sometimes appear,

$$\sigma(p,\nu)=\sum_{k=1}^{\infty}\frac{1}{\xi_{\nu k}^{2p}}, \tag{1}$$

most often for $p=1$, where $\xi_{\nu k}$ are the positions of the zeros located away from the origin of the regular cylindrical Bessel function $J_\nu(\xi)$, with real $\nu\geq 0$ and integer $p>0$. The sums are convergent for $p\geq 1$. As we will show in what follows, all these sums have the property that they are given by the ratio of two polynomials on $\nu$ with integer coefficients. The simplest and most common example is

$$\sigma(1,\nu)=\frac{1}{4(\nu+1)}. \tag{2}$$

In a few cases the exact expression of these polynomials are available in the literature [1]. The known cases are those obtained by Rayleigh, extending investigations by Euler, for $p=1$ through $p=5$, and one discovered by Cayley, for $p=8$. The cases $p=6$ and $p=7$ seem not to be generally known, and will be given explicitly further on. The known cases were obtained in a case-by-case fashion, using the expression of the Bessel functions as infinite products involving its zeros.

In this paper we will provide a simple, independent proof of all the known formulas, and will present a general formula from which the specific formulas can be derived, for any given strictly positive integer value of $p$, by purely algebraic means. The proof will rely entirely on the general properties of analytical functions and on the well-known properties of the functions $J_\nu(\xi)$, which are generally available in the literature, for example in [2].

## 2 Definition of the Elements Involved

The proof will be based on the singularity structure of the following analytical function in the complex-$\xi$ plane,

$$f(p, \nu, \xi) = \frac{J_{\nu+p}(\xi)}{\xi^{p+1} J_\nu(\xi)}, \tag{3}$$

where for the time being we may consider that $\nu \geq 0$ and $p > 0$ are real numbers. In case any of the functions involved have branching points at $\xi = 0$, we consider the cuts to be over the negative real semi-axis. Preliminary to the proof, it will be necessary to establish a few properties of this function.

We will consider the contour integral of this function over the circuit on the complex-$\xi$ plane shown in Fig. 1, in the $R \to \infty$ limit. Since this circuit goes through the origin $\xi = 0$, where the function will be seen to have a simple pole, we will adopt for the integral the principal value of Cauchy. The limit $R \to \infty$ will be taken in a discrete way, in order to avoid going through the other singularities of the function, which are located at $\xi = \xi_{\nu k}$. We will see that, for large values of $R$ and $j$, it is possible to adopt for $R$ the values given by

$$R = \pi \, \frac{2\nu + 1}{4} + j\pi, \tag{4}$$

where for each $k$ there is a value of the integer $j$ such that $R$ is strictly within the interval $(\xi_{\nu k}, \xi_{\nu(k+1)})$. In this way each step in the discrete $R \to \infty$ limit will correspond to a partial sum of the infinite sums involved.

The proof of the expressions for $\sigma(p, \nu)$ consists of two parts: first, the proof that the integral of $f(p, \nu, \xi)$ over the circuit is zero in the $R \to \infty$ limit, for all $p > 0$ and all $\nu \geq 0$; second, the use of the residue theorem. This will result in a general formula from which the expressions for the sums $\sigma(p, \nu)$ can be derived. We will also present a partial solution of the problem of deriving the formulas for $\sigma(p, \nu)$, which will take the form of another general formula from which these expressions can be derived algebraically.

## 3 Properties of $f(p, \nu, \xi)$

Let us establish a few important properties of $f(p, \nu, \xi)$, starting by its behavior under the inversion of the sign of $\xi$. We start with the analogous property of $J_\nu(\xi)$, for which we have, using the Maclaurin series for these functions [3], which converges over the whole complex plane,

$$J_\nu(-\xi) = (-1)^\nu J_\nu(\xi). \tag{5}$$

Using this in the expression for $f(p, \nu, \xi)$ we get

$$f(p, \nu, -\xi) = -f(p, \nu, \xi), \tag{6}$$

that is, $f(p, \nu, \xi)$ is an odd function of $\xi$, for all $p$ and all $\nu$.

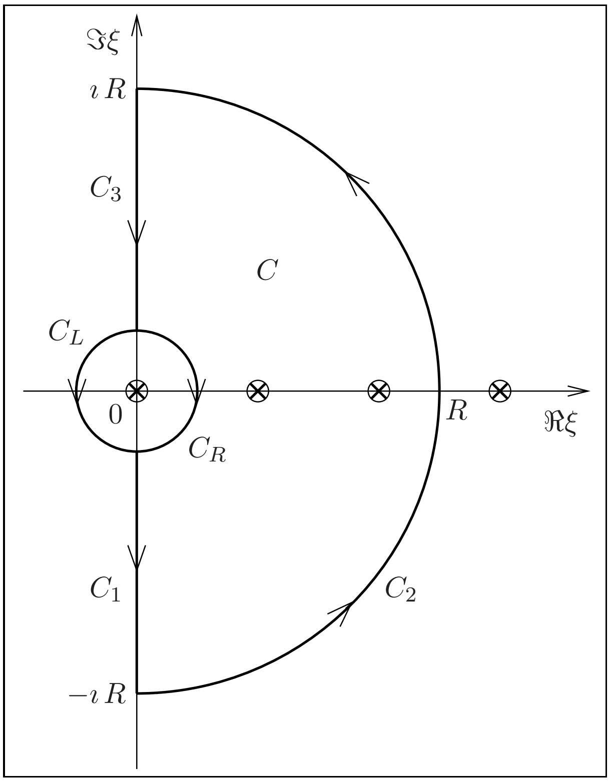


Figure 1: The integration contour in the complex-$\xi$ plane, showing the various parts of the circuit $C$ and some of the singularities of $f(p,\nu,\xi)$.

Next we show that $f(p,\nu,\xi)$ has a simple pole at $\xi = 0$. Since $J_{\nu+p}(\xi)$, $\xi^{p+1}$ and $J_\nu(\xi)$ are analytical functions over the whole complex-$\xi$ plane, it follows that $f(p,\nu,\xi)$ is analytical over the whole plane except for those points where the denominator vanishes, where it has poles. These are the origin $\xi = 0$ and the zeros $\xi_{\nu k}$ of the Bessel function in the denominator. Note that while for non-integer $\nu$ and $p$ the functions involved have branching points at $\xi = 0$, the function $f(p,\nu,\xi)$ never does. In order to determine the residue of $f(p,\nu,\xi)$ at $\xi = 0$ we consider the limit

$$\begin{aligned} r_0(p,\nu) &= \lim_{\xi\to 0} \xi f(p,\nu,\xi) \\ &= \frac{\Gamma(\nu+1)}{2^p\Gamma(\nu+p+1)}, \end{aligned} \tag{7}$$

where $\Gamma(z)$ is the gamma function and we used once more the Maclaurin series for $J_\nu(\xi)$. Since the limit is finite and non-zero, it follows that $f(p,\nu,\xi)$ has a simple pole at $\xi = 0$, and that $r_0(p,\nu)$ is the corresponding residue. Turning to the poles at $\xi_{\nu k}$, since $\xi_{\nu k}$ is a simple zero of $J_\nu(\xi)$, at which its derivative $J'_\nu(\xi)$ is different from zero, it follows that $f(p,\nu,\xi)$ has a Taylor expansion around this point, with the form

$$J_\nu(\xi) = (\xi - \xi_{\nu k})J'_\nu(\xi_{\nu k}) + \sum_{i=2}^{\infty} c_i(\xi - \xi_{\nu k})^i, \tag{8}$$

for certain finite coefficients $c_i$. In order to determine the residue of $f(p,\nu,\xi)$ at $\xi_{\nu k}$ we consider then the limit

$$
\begin{aligned}
r_k(p,\nu) &= \lim_{\xi\to\xi_{\nu k}} (\xi-\xi_{\nu k}) f(p,\nu,\xi) \\
&= \frac{J_{\nu+p}(\xi_{\nu k})}{\xi_{\nu k}^{p+1}\, J'_\nu(\xi_{\nu k})},
\end{aligned}
\tag{9}
$$

where we used this Taylor expansion. Since the derivative is finite and non-zero at $\xi_{\nu k}$, this limit also is finite and non-zero, and hence it follows that $f(p,\nu,\xi)$ has a simple pole at $\xi_{\nu k}$, and that $r_k(p,\nu)$ is the corresponding residue. We can simplify this expression using the well-known identity [4]

$$
\xi \frac{\partial}{\partial\xi} J_\nu(\xi) = -\xi J_{\nu+1}(\xi) + \nu J_\nu(\xi),
\tag{10}
$$

which applied at $\xi=\xi_{\nu k}$, since $J_\nu(\xi_{\nu k})=0$, results in

$$
J'_\nu(\xi_{\nu k}) = -J_{\nu+1}(\xi_{\nu k}).
\tag{11}
$$

It follows therefore that we have for the residues of the poles at $\xi_{\nu k}$,

$$
r_k(p,\nu) = -\frac{J_{\nu+p}(\xi_{\nu k})}{\xi_{\nu k}^{p+1}\, J_{\nu+1}(\xi_{\nu k})}.
\tag{12}
$$

We will now establish the behavior of the absolute value of the ratio of Bessel functions which is contained in the expression of $f(p,\nu,\xi)$ in Eq. (3), for large values of $R$. In order to do this we use the asymptotic expansion of the Bessel functions [5], valid in the whole complex plane so long as $\arg(\xi)\neq\pm\pi$, written in terms of $R=|\xi|$ and $\theta=\arg(\xi)$, to the lowest orders, and with the trigonometric functions expressed as complex exponentials,

$$
\begin{aligned}
J_\nu(\xi) &= \sqrt{\frac{e^{-\imath\theta}}{2\pi R}} \left\{ \left[1+\frac{\mathcal{R}_c(\nu,\xi)}{R^2\, e^{\imath 2\theta}}\right] \times \right. \\
&\qquad \times \left[e^{-R\sin(\theta)}\, e^{\imath\,\alpha(R,\theta,\nu)} + e^{R\sin(\theta)}\, e^{-\imath\,\alpha(R,\theta,\nu)}\right] + \\
&\qquad +\imath \left[\frac{4\nu^2-1}{8R\, e^{\imath\theta}} + \frac{\mathcal{R}_s(\nu,\xi)}{R^3\, e^{\imath 3\theta}}\right] \times \\
&\qquad \left. \times \left[e^{-R\sin(\theta)}\, e^{\imath\,\alpha(R,\theta,\nu)} - e^{R\sin(\theta)}\, e^{-\imath\,\alpha(R,\theta,\nu)}\right] \right\},
\end{aligned}
\tag{13}
$$

where $\mathcal{R}_c(\nu,\xi)$ and $\mathcal{R}_s(\nu,\xi)$ are certain limited functions of $\xi$ and $\alpha(R,\theta,\nu)$ is a certain real number, given by

$$
\alpha(R,\theta,\nu) = R\cos(\theta) - \pi\,\frac{2\nu+1}{4}.
\tag{14}
$$

The behavior of the expression in Eq. (13) for large values of $R$ depends on the sign of $\theta$, and the particular case $\theta=0$ has to be examined separately. In this particular case we have

$$
\begin{aligned}
J_\nu(\xi) &= \sqrt{\frac{2}{\pi R}} \left\{ \left[1+\frac{\mathcal{R}_c(\nu,\xi)}{R^2}\right] \cos[\alpha(R,0,\nu)] + \right. \\
&\qquad \left. - \left[\frac{4\nu^2-1}{8R} + \frac{\mathcal{R}_s(\nu,\xi)}{R^3}\right] \sin[\alpha(R,0,\nu)] \right\},
\end{aligned}
\tag{15}
$$

where all the functions involved are now limited, so that for large values of $R$ we have for the dominant part of $J_\nu(\xi)$,

$$J_\nu(\xi) \approx \sqrt{\frac{2}{\pi R}}\,\cos[\alpha(R,0,\nu)]. \tag{16}$$

Note now that the points where $\cos[\alpha(R,0,\nu)] = 0$ are the zeros of $J_\nu(\xi)$, expressed in the asymptotic limit. We will now choose a way to take the $R \to \infty$ limit such that these zeros are avoided. We may simply chose for the passage of the circuit across the real axis that point between two zeros where $\cos[\alpha(R,0,\nu)] = \pm 1$ and $\sin[\alpha(R,0,\nu)] = 0$. Since for $\theta = 0$ we have

$$\alpha(R,0,\nu) = R - \pi\,\frac{2\nu+1}{4}, \tag{17}$$

and we must have $\alpha(R,0,\nu) = j\pi$ for some integer $j$, we conclude that Eq. (4) holds, which will cause the crossing of the circuit and the real axis to avoid the zeros. This defines the $R \to \infty$ limit in full detail. It follows that for our purposes here we may write the asymptotic expansion in the case $\theta = 0$ as

$$J_\nu(\xi) = \pm\sqrt{\frac{2}{\pi R}}\left[1 + \frac{\mathcal{R}_c(\nu,\xi)}{R^2}\right]. \tag{18}$$

In the case $\theta > 0$ we put the dominant real exponential in evidence and obtain

$$\begin{aligned}
J_\nu(\xi) \;=\; \sqrt{\frac{e^{-\imath\theta}}{2\pi R}}\,e^{R\sin(\theta)} \Bigg\{ & \left[1 + \frac{\mathcal{R}_c(\nu,\xi)}{R^2\,e^{\imath 2\theta}}\right] \times \\
& \times \left[e^{-2R\sin(\theta)}\,e^{\imath\,\alpha(R,\theta,\nu)} + e^{-\imath\,\alpha(R,\theta,\nu)}\right] + \\
& +\imath \left[\frac{4\nu^2-1}{8R\,e^{\imath\theta}} + \frac{\mathcal{R}_s(\nu,\xi)}{R^3\,e^{\imath 3\theta}}\right] \times \\
& \times \left[e^{-2R\sin(\theta)}\,e^{\imath\,\alpha(R,\theta,\nu)} - e^{-\imath\,\alpha(R,\theta,\nu)}\right] \Bigg\},
\end{aligned} \tag{19}$$

where all the functions within the brackets are now limited or go to zero in the $R \to \infty$ limit. Finally, we do the same thing for the case $\theta < 0$, obtaining

$$\begin{aligned}
J_\nu(\xi) \;=\; \sqrt{\frac{e^{-\imath\theta}}{2\pi R}}\,e^{-R\sin(\theta)} \Bigg\{ & \left[1 + \frac{\mathcal{R}_c(\nu,\xi)}{R^2\,e^{\imath 2\theta}}\right] \times \\
& \times \left[e^{\imath\,\alpha(R,\theta,\nu)} + e^{2R\sin(\theta)}\,e^{-\imath\,\alpha(R,\theta,\nu)}\right] + \\
& +\imath \left[\frac{4\nu^2-1}{8R\,e^{\imath\theta}} + \frac{\mathcal{R}_s(\nu,\xi)}{R^3\,e^{\imath 3\theta}}\right] \times \\
& \times \left[e^{\imath\,\alpha(R,\theta,\nu)} - e^{2R\sin(\theta)}\,e^{-\imath\,\alpha(R,\theta,\nu)}\right] \Bigg\},
\end{aligned} \tag{20}$$

where once more all the functions within the brackets are now limited or go to zero in the $R \to \infty$ limit. We are now in a position to analyze the behavior of the absolute value of the ratio of two Bessel functions which appears in the definition of $f(p,\nu,\xi)$. The factors which do not depend on $\nu$ are common to the numerator and denominator, and cancel out. In the case $\theta = 0$ we get

$$\left|\frac{J_{\nu+p}(\xi)}{J_\nu(\xi)}\right| = \left|\frac{1 + \dfrac{\mathcal{R}_c(\nu+p,\xi)}{R^2}}{1 + \dfrac{\mathcal{R}_c(\nu,\xi)}{R^2}}\right|, \tag{21}$$

so that in the $R \to \infty$ limit we get

$$\lim_{R\to\infty} \left| \frac{J_{\nu+p}(\xi)}{J_\nu(\xi)} \right| = 1. \tag{22}$$

It is not difficult to verify that for both the case $\theta > 0$ and the case $\theta < 0$ we get this same value for this limit. We see therefore that the $R \to \infty$ limit of the absolute value of this ratio is simply 1, for all values of $\theta$ in $(-\pi, \pi)$.

## 4 Evaluation of the Integral

Let us consider now the proof that the integral is zero. In order to do this we will separate the circuit in sections and prove the result for each section. The complete circuit $C$ consists of two straight sections $C_1$ e $C_3$, of the great semicircle $C_2$ and of two small semicircles $C_L$ e $C_R$ of radius $\varepsilon$ around the point $\xi = 0$.

For the pair of straight lines $C_1$ e $C_3$, where we have $d\xi = \imath\, dy$ with $\xi = x + \imath\, y$, taking into account the orientation, we may write

$$\begin{aligned} I_{C_1+C_3} &= \int_{C_1+C_3} f(p,\nu,\xi)\, d\xi \\ &= \imath \int_R^{\varepsilon} f(p,\nu,\xi)\, dy + \imath \int_{-\varepsilon}^{-R} f(p,\nu,\xi)\, dy. \end{aligned} \tag{23}$$

Making in the second integral the transformation of variables $\xi \to -\xi$, which implies $x \to -x$ e $y \to -y$, and since $f(p,\nu,\xi)$ is odd, we have

$$I_{C_1+C_3} = 0. \tag{24}$$

We see therefore that this part of the integral vanishes exactly, independently of the values of $R$ and $\varepsilon$. We are therefore free to take limits involving $R$ or $\varepsilon$ during the calculation of the other sections of the integral, without affecting this result.

Next we consider the two semicircles of radius $\varepsilon$. We will denote this part of the integral, to be calculated according to the criterion of the principal value of Cauchy, as

$$I_{C_L+C_R} = \int_{C_L+C_R} f(p,\nu,\xi)\, d\xi. \tag{25}$$

Since the function $f(p,\nu,\xi)$ has a simple pole at $\xi = 0$, and is also odd, it can be expressed as a Laurent series around this point, with the form

$$f(p,\nu,\xi) = \frac{r_0(p,\nu)}{\xi} + \sum_{i=0}^{\infty} c_i \xi^{2i+1}, \tag{26}$$

where $r_0(p,\nu)$ is the residue of the function at this point, and $c_i$ are certain finite coefficients. The series is convergent so long as $\varepsilon$ is smaller than the first zero $\xi_{\nu 1}$. The sum of positive powers represents an analytical function around $\xi = 0$, and is therefore regular within the circle of radius $\varepsilon$. It follows that the integral of this regular part goes to zero in the limit $\varepsilon \to 0$, since in this limit both each individual term of the sum and the measure of the domain of integration vanish.

It follows that only the integral of the term containing the pole can remain different from zero in the $\varepsilon \to 0$ limit. We will therefore calculate this integral in polar coordinates, with $\xi = \varepsilon \exp(\imath \theta)$ and $d\xi = \imath \varepsilon \exp(\imath \theta) d\theta$. Since we must use here the Cauchy principal value we have for this part $\Delta I_{C_L+C_R}$ of the integral $I_{C_L+C_R}$,

$$\begin{aligned}
\Delta I_{C_L+C_R} &= \frac{1}{2} \int_{C_L} \frac{r_0(p,\nu)}{\xi}\, d\xi + \frac{1}{2} \int_{C_R} \frac{r_0(p,\nu)}{\xi}\, d\xi \\
&= \frac{\imath\, r_0(p,\nu)}{2} \int_{\pi/2}^{3\pi/2} d\theta + \frac{\imath\, r_0(p,\nu)}{2} \int_{\pi/2}^{-\pi/2} d\theta \\
&= 0.
\end{aligned} \tag{27}$$

Therefore, this part of the integral $I_{C_L+C_R}$ also vanishes, and hence the integral $I_{C_L+C_R}$ vanishes in the limit $\varepsilon \to 0$. Since during this deformation of the circuit no singularities of the function are crossed, and hence the integral does not change, if follows that the integral is zero for all values of $\varepsilon$ smaller than $\xi_{\nu 1}$.

The last section of the circuit we must consider is $C_2$. In this case the integral is not zero for finite values of $R$, but we may show that it goes to zero in the limit $R \to \infty$, subject to the condition that for large values of $R$ we have that Eq. (4) holds, so that the circuit does not go over any of the singularities at the points $\xi_{\nu k}$. Using once more polar coordinates, in this section of the circuit we have $d\xi = \imath\, R \exp(\imath \theta) d\theta$, where $\xi = R \exp(\imath \theta)$, so that the integral is given by

$$\begin{aligned}
I_{C_2} &= \int_{C_2} f(p,\nu,\xi)\, d\xi \\
&= \imath\, R \int_{-\pi/2}^{\pi/2} d\theta\, e^{\imath \theta}\, f(p,\nu,\xi).
\end{aligned} \tag{28}$$

Taking the absolute value of the integral and using the triangle inequalities we have

$$|I_{C_2}| \leq R \int_{-\pi/2}^{\pi/2} d\theta\, |f(p,\nu,\xi)|, \tag{29}$$

for any value of $R$, and hence also in the $R \to \infty$ limit. We must now consider the behavior of the absolute value of $f(p,\nu,\xi)$ for large values of $R$. In order to do this we calculate the limit

$$\begin{aligned}
|I_{C_2}| &\leq \lim_{R\to\infty} \left[ R \int_{-\pi/2}^{\pi/2} d\theta\, |f(p,\nu,\xi)| \right] \\
&= \lim_{R\to\infty} \left[ R \int_{-\pi/2}^{\pi/2} d\theta\, \frac{1}{R^{p+1}} \left| \frac{J_{\nu+p}(\xi)}{J_\nu(\xi)} \right| \right].
\end{aligned} \tag{30}$$

As we established before, the limit of the absolute value of the ratio of the two Bessel functions is 1. As a consequence of this, we have for the integral over the section $C_2$ of the circuit, in the $R \to \infty$ limit,

$$\begin{aligned}
|I_{C_2}| &\leq \lim_{R\to\infty} \left[ \int_{-\pi/2}^{\pi/2} d\theta\, \frac{1}{R^p} \right] \\
&= 0,
\end{aligned} \tag{31}$$

since we have $p > 0$. This implies, of course, that $I_{C_2} = 0$ in the $R \to \infty$ limit. We see therefore that the integral of $f(p,\nu,\xi)$ over the circuit $C$, in the $R \to \infty$ limit, vanishes in all sections of the circuit, and hence that the integral is zero in the $R \to \infty$ limit,

$$\lim_{R\to\infty} \oint_C f(p,\nu,\xi)\, d\xi = 0. \tag{32}$$

## 5 Using the Residue Theorem

Considering that in the $R \to \infty$ limit the poles with residues $r_k(p,\nu)$ are all that exist strictly within the circuit, that the pole with residue $r_0(p,\nu)$ is the only one located over the circuit, and that the integral is defined as the Cauchy principal value at this pole, we can use the residue theorem to write for the integral

$$\lim_{R\to\infty} \oint_C f(p,\nu,\xi)\,d\xi = 2\pi\imath \left[\frac{1}{2}\,r_0(p,\nu) + \sum_{k=1}^{\infty} r_k(p,\nu)\right]. \tag{33}$$

On the other hand, as we saw above the integral vanishes in the $R \to \infty$ limit, and hence we have

$$\frac{1}{2}\,r_0(p,\nu) + \sum_{k=1}^{\infty} r_k(p,\nu) = 0. \tag{34}$$

We have therefore the following general result involving all these residues, substituting the values we calculated before for each one of them,

$$\frac{\Gamma(\nu+1)}{2^{p+1}\Gamma(\nu+p+1)} = \sum_{k=1}^{\infty} \frac{1}{\xi_{\nu k}^{p+1}} \frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}. \tag{35}$$

This is valid for any real value of $\nu \geq 0$ and for any real value of $p > 0$.

## 6 Proof of Some Known Formulas

Up to this point $p$ could be any strictly positive real number. From now on, however, we have to assume that $p$ is a strictly positive *integer*. In order to further simplify the expression obtained above, in general it will be necessary to write $J_{\nu+p}(\xi_{\nu k})$ in terms of $J_{\nu+1}(\xi_{\nu k})$, which can be done using the recurrence formula of the Bessel functions [6], so long as $p$ is an integer. Let us examine a few of the initial cases. For $p = 1$ we have simply

$$\frac{\Gamma(\nu+1)}{2^2\Gamma(\nu+2)} = \sum_{k=1}^{\infty} \frac{1}{\xi_{\nu k}^2}, \tag{36}$$

so that the formula for the sum that corresponds to this case is

$$\begin{aligned} \sigma(1,\nu) &= \sum_{k=1}^{\infty} \frac{1}{\xi_{\nu k}^2} \\ &= \frac{1}{2^2(\nu+1)}, \end{aligned} \tag{37}$$

were we used the properties of the gamma function, thus obtaining a polynomial on $\nu$ in the denominator. In this way we obtain the first of the known results, and this formula is therefore proven, being valid for any non-negative real value of $\nu$. For $p = 2$ we have

$$\frac{\Gamma(\nu+1)}{2^3\Gamma(\nu+3)} = \sum_{k=1}^{\infty} \frac{1}{\xi_{\nu k}^3} \frac{J_{\nu+2}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}. \tag{38}$$

In order to simplify this expression we write the recurrence formula as

$$J_{\nu+2}(\xi) = \frac{2(\nu+1)}{\xi} J_{\nu+1}(\xi) - J_\nu(\xi), \tag{39}$$

where we exchanged $\nu$ for $\nu + 1$. Applying this for $\xi = \xi_{\nu k}$ and using once more the fact that $J_\nu(\xi_{\nu k}) = 0$, we get

$$J_{\nu+2}(\xi_{\nu k}) = \frac{2(\nu+1)}{\xi_{\nu k}} J_{\nu+1}(\xi_{\nu k}), \tag{40}$$

so that in this case we have

$$\frac{\Gamma(\nu+1)}{2^3\Gamma(\nu+3)} = 2(\nu+1)\sum_{k=1}^{\infty}\frac{1}{\xi_{\nu k}^4}, \tag{41}$$

from which it follows that the formula for the sum that corresponds to this case is

$$\begin{aligned} \sigma(2,\nu) &= \sum_{k=1}^{\infty}\frac{1}{\xi_{\nu k}^4} \\ &= \frac{1}{2^4(\nu+1)^2(\nu+2)}. \end{aligned} \tag{42}$$

We thus obtain the second known result, which is now proven, and which also has a polynomial on $\nu$ in the denominator. The proof of the first two formulas is therefore quite straightforward. In the $p = 3$ case, however, something slightly different happens. In this case we have

$$\frac{\Gamma(\nu+1)}{2^4\Gamma(\nu+4)} = \sum_{k=1}^{\infty}\frac{1}{\xi_{\nu k}^4}\,\frac{J_{\nu+3}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}, \tag{43}$$

and hence we must write one more version of the recurrence formula. Exchanging $\nu$ for $\nu + 2$ in the original formula, and applying at $\xi = \xi_{\nu k}$, we obtain

$$J_{\nu+3}(\xi_{\nu k}) = \frac{2(\nu+2)}{\xi_{\nu k}} J_{\nu+2}(\xi_{\nu k}) - J_{\nu+1}(\xi_{\nu k}). \tag{44}$$

Substituting in this the solution found for the previous case, which gives us $J_{\nu+2}(\xi_{\nu k})$ in terms of $J_{\nu+1}(\xi_{\nu k})$, we get

$$J_{\nu+3}(\xi_{\nu k}) = \left[\frac{2^2(\nu+1)(\nu+2)}{\xi_{\nu k}^2} - 1\right] J_{\nu+1}(\xi_{\nu k}). \tag{45}$$

In this way we get in this case the result

$$\begin{aligned} \frac{\Gamma(\nu+1)}{2^4\Gamma(\nu+4)} &= \sum_{k=1}^{\infty}\frac{1}{\xi_{\nu k}^4}\left[\frac{2^2(\nu+1)(\nu+2)}{\xi_{\nu k}^2} - 1\right] \\ &= 2^2(\nu+1)(\nu+2)\sigma(3,\nu) - \sigma(2,\nu). \end{aligned} \tag{46}$$

We see that in this case a linear combination of the sums of two different powers of the zeros $\xi_{\nu k}$ appears. Using the properties of the gamma function and substituting the value obtained previously for $\sigma(2,\nu)$, we get for $\sigma(3,\nu)$

$$\sigma(3,\nu) = \frac{2(\nu+2)}{2^6(\nu+1)^3(\nu+2)^2(\nu+3)}. \tag{47}$$

The formula for the sum that corresponds to this case is therefore

$$\sigma(3,\nu) = \sum_{k=1}^{\infty} \frac{1}{\xi_{\nu k}^{6}} = \frac{1}{2^5(\nu+1)^3(\nu+2)(\nu+3)}, \tag{48}$$

where we once more have a polynomial on $\nu$ in the denominator. We thus obtain the third of the known results, and the formula for $\sigma(3,\nu)$ is proven.

It is clear that we can proceed in this way indefinitely, thus obtaining the formulas for successive values of $p$. In each case it is necessary to first use the recurrence formula in order to write $J_{\nu+p}(\xi)$ in terms of $J_{\nu+1}(\xi)$. In general the result will be a linear combination of sums of several distinct inverse powers of $\xi_{\nu k}$. At this point the use of the general formula in Eq. (35) will produce an expression for the linear combination of the corresponding sums $\sigma(p,\nu)$. Finally, it is necessary to solve the resulting expression for the sum with the largest value of $p$ so far, using for this end the results obtained previously for the other sums. In this way all the formulas for the sums $\sigma(p,\nu)$ can be derived successively by purely algebraic means, resulting every time in the ratio of two polynomials, with the one in denominator completely factored.

## 7 Proof of a General Formula

It is possible to systematize the resolution process described above to the point where a general formula for the linear combination of the sums $\sigma(p,\nu)$ can be written. This is based on a systematization of the general formula for the ratio of Bessel functions, which is found to be

$$\frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} = \sum_{q=0}^{q_M} (-1)^q \frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)} \left(\frac{2}{\xi_{\nu k}}\right)^{(p-1)-2q}, \tag{49}$$

where $q_M = (p-1)/2$ for odd $p$ and $q_M = (p-2)/2$ for even $p$, and for which we will provide proof in what follows. The use of the general formula in Eq. (35) then produces a corresponding general formula for the linear combination of the sums $\sigma(p,\nu)$,

$$\frac{\Gamma(\nu+1)}{2^p\Gamma(\nu+p+1)} = \sum_{q=0}^{q_M} (-1)^q\, 2^{p-2q} \frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)}\, \sigma(p-q,\nu). \tag{50}$$

Note that the left-hand side of this equation can be written as the inverse of a polynomial on $\nu$, with integer coefficients, with the simple use of the properties of the gamma function. On the other hand, the coefficients on the right-hand side can all be written as polynomials on $\nu$, with integer coefficients, since we have for the arguments of the two gamma functions, in the numerator and in the denominator,

$$(\nu+p-q) = (\nu+q+1) + p - 2q - 1, \tag{51}$$

where $p-2q-1$ is an integer whose minimum value is 0 for odd $p$, and 1 for even $p$. It follows that, once the equation is solved for $\sigma(p,\nu)$, resulting in

$$\frac{\Gamma(\nu+p)}{\Gamma(\nu+1)}\sigma(p,\nu) \;=\; \frac{\Gamma(\nu+1)}{2^{2p}\Gamma(\nu+p+1)}+ \\ -\sum_{q=1}^{q_M}\frac{(-1)^q}{2^{2q}}\frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\;q!\;\Gamma(\nu+q+1)}\sigma(p-q,\nu), \qquad (52)$$

and assuming that the previous sums all have this same property, the expression for this sum will have the form of the ratio of two polynomials on $\nu$, with integer coefficients. Hence, since we saw that this is valid for the first three sums, by finite induction it is valid for *all* the sums.

This set of equations, taken for all strictly positive integer values of $p$, forms an infinite linear system of equations in triangular form, that can be solved iteratively in order to obtain closed forms for $\sigma(p,\nu)$ in a purely algebraic way, in principle for arbitrary integer values of $p$, although for large values of $p$ the algebraic work involved can be very large. However, it is straight, direct algebraic work, well suited for a computer-algebra approach.

We will now prove these two general formulas. Since the general formula in Eq. (50) follows from the general formula in Eq. (49), it suffices to prove the latter. We can do this by finite induction. Since the upper limits of the summations involved depend on the parity of $p$, it is necessary to consider the two cases separately. The first step is to verify that our general formula reproduces the correct results for the first three cases, which we have already derived individually. Applying the general formula in Eq. (49) for $p=1$, in which case we have $q_M=0$, we obtain at once

$$\frac{J_{\nu+1}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}=1, \qquad (53)$$

which is obviously the correct result. Applying now the same general formula for $p=2$, for which we also have $q_M=0$, we get

$$\frac{J_{\nu+2}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}=\frac{2(\nu+1)}{\xi_{\nu k}}, \qquad (54)$$

which is also the correct result. Finally, applying the general formula for $p=3$, in which case we have $q_M=1$, we obtain

$$\frac{J_{\nu+3}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}=\frac{2^2(\nu+2)(\nu+1)}{\xi_{\nu k}^2}-1, \qquad (55)$$

which once more is the correct result. It suffices now to use the recurrence formula of the Bessel functions to show that the formula for $p$ follows from the previous formulas, for $p-1$ e $p-2$. We start with the case in which $p$ is even, and writing explicitly the upper limits of the sums, we have

$$\frac{J_{\nu+p-1}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} \;=\; \sum_{q=0}^{(p-2)/2}(-1)^q\frac{[(p-2)-q]!\,\Gamma(\nu+p-1-q)}{[(p-2)-2q]!\;q!\;\Gamma(\nu+q+1)}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-2)-2q}, \qquad (56)$$

$$\frac{J_{\nu+p-2}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} \;=\; \sum_{q=0}^{(p-4)/2}(-1)^q\frac{[(p-3)-q]!\,\Gamma(\nu+p-2-q)}{[(p-3)-2q]!\;q!\;\Gamma(\nu+q+1)}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-3)-2q}. \qquad (57)$$

Writing now the recurrence formula which gives the function $J_{\nu+p}(\xi_{\nu k})$ in terms of $J_{\nu+p-1}(\xi_{\nu k})$ and $J_{\nu+p-2}(\xi_{\nu k})$, and substituting Eq. (56) and (57), we get, after some manipulation of the indices of the sums,

$$\begin{aligned}
J_{\nu+p}(\xi_{\nu k}) &= \frac{2(\nu+p-1)}{\xi_{\nu k}}\,J_{\nu+p-1}(\xi_{\nu k})-J_{\nu+p-2}(\xi_{\nu k})\Rightarrow\\
\frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} &= \frac{2(\nu+p-1)}{\xi_{\nu k}}\,\frac{J_{\nu+p-1}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}-\frac{J_{\nu+p-2}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}\\
&= \sum_{q=0}^{(p-2)/2}(-1)^q\,\frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)}\times\\
&\qquad\times\frac{(\nu+p-1)[(p-1)-2q]+q(\nu+q)}{[(p-1)-q][\nu+(p-1)-q]}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-1)-2q}. \qquad (58)
\end{aligned}$$

It is easy to verify that we have for the second fraction in this sum,

$$\frac{(\nu+p-1)[(p-1)-2q]+q(\nu+q)}{[(p-1)-q][\nu+(p-1)-q]}=1. \qquad (59)$$

We therefore conclude that

$$\frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}=\sum_{q=0}^{(p-2)/2}(-1)^q\,\frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-1)-2q}, \qquad (60)$$

thus proving the general formula for even $p$. For odd $p$, once more writing explicitly the upper limit of the sums, we start from

$$\begin{aligned}
\frac{J_{\nu+p-1}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} &= \sum_{q=0}^{(p-3)/2}(-1)^q\,\frac{[(p-2)-q]!\,\Gamma(\nu+p-1-q)}{[(p-2)-2q]!\,q!\,\Gamma(\nu+q+1)}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-2)-2q}, \qquad (61)\\
\frac{J_{\nu+p-2}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} &= \sum_{q=0}^{(p-3)/2}(-1)^q\,\frac{[(p-3)-q]!\,\Gamma(\nu+p-2-q)}{[(p-3)-2q]!\,q!\,\Gamma(\nu+q+1)}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-3)-2q}. \qquad (62)
\end{aligned}$$

Writing once again the recurrence formula which gives $J_{\nu+p}(\xi_{\nu k})$ in terms of $J_{\nu+p-1}(\xi_{\nu k})$ and $J_{\nu+p-2}(\xi_{\nu k})$, and substituting Eq. (61) and (62), we get, after some similar manipulation of the indices of the sums,

$$\begin{aligned}
J_{\nu+p}(\xi_{\nu k}) &= \frac{2(\nu+p-1)}{\xi_{\nu k}}\,J_{\nu+p-1}(\xi_{\nu k})-J_{\nu+p-2}(\xi_{\nu k})\Rightarrow\\
\frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} &= \frac{2(\nu+p-1)}{\xi_{\nu k}}\,\frac{J_{\nu+p-1}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}-\frac{J_{\nu+p-2}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})}\\
&= \sum_{q=0}^{(p-3)/2}(-1)^q\,\frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)}\times\\
&\qquad\times\frac{(\nu+p-1)[(p-1)-2q]+q(\nu+q)}{[(p-1)-q][\nu+(p-1)-q]}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-1)-2q}\\
&\quad+(-1)^{(p-1)/2}. \qquad (63)
\end{aligned}$$

The second fraction within the last summation, which does not involve factorials, is the same as before, and therefore is equal to 1. It follows that we have

$$\begin{aligned}
\frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} &= \sum_{q=0}^{(p-3)/2}(-1)^q\,\frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)}\left(\frac{2}{\xi_{\nu k}}\right)^{(p-1)-2q}\\
&\quad+(-1)^{(p-1)/2}. \qquad (64)
\end{aligned}$$

It is not difficult to verify that the additional term that we have here is in fact equal to the argument of the summation in the case $q = (p-1)/2$, so that we may merge it with the summation and this obtain

$$\frac{J_{\nu+p}(\xi_{\nu k})}{J_{\nu+1}(\xi_{\nu k})} = \sum_{q=0}^{(p-1)/2} (-1)^q \frac{[(p-1)-q]!\,\Gamma(\nu+p-q)}{[(p-1)-2q]!\,q!\,\Gamma(\nu+q+1)} \left(\frac{2}{\xi_{\nu k}}\right)^{(p-1)-2q}, \qquad (65)$$

thus proving the general formula in this case. This completes the proof of the general formula in Eq. (49), from which follows the general formula in Eq. (50) for the linear combination of the sums $\sigma(p,\nu)$, which is therefore proven as well.

# 8 Some Particular Cases

We may now use the general formula in Eq. (50) for $\sigma(p,\nu)$ in order to write explicitly a few cases which are not found in the current literature. With a little help from the free-software algebraic manipulation program `maxima`, we get the following two results, thus completing the sequence of known results up to $p = 8$,

$$\sigma(6,\nu) = \frac{21\nu^3 + 181\nu^2 + 513\nu + 473}{2^{11}(\nu+1)^6(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)}, \qquad (66)$$

$$\sigma(7,\nu) = \frac{33\nu^3 + 329\nu^2 + 1081\nu + 1145}{2^{12}(\nu+1)^7(\nu+2)^3(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)(\nu+7)}. \qquad (67)$$

We now point out that our results for $\sigma(p,\nu)$ are valid for all real values of $\nu$, not just for the integers. Therefore, exchanging $\nu$ for $\nu + 1/2$ we may obtain formulas that are valid for the zeros of the regular spherical Bessel functions $j_\nu(\xi)$, since we have the well-known relation

$$j_\nu(\xi) = \sqrt{\frac{\pi}{2\xi}}\, J_{\nu+1/2}(\xi) \qquad (68)$$

between these two families of functions. In particular, using the value $\nu = 1/2$ we obtain the results for $j_0(\xi)$, whose zeros are given by $k\pi$, since this particular function is proportional to $\sin(\xi)$ [7]. In this way we obtain a direct relation between our results and the Riemann zeta function, for certain real integer arguments of $\zeta(z)$. In fact, we have

$$\begin{aligned} \sigma(p,1/2) &= \sum_{k=1}^{\infty} \frac{1}{\pi^{2p}k^{2p}} \Rightarrow \\ \zeta(2p) &= \pi^{2p}\sigma(p,1/2). \end{aligned} \qquad (69)$$

Using the formulas we obtained here for $\sigma(p,\nu)$ in the case $\nu = 1/2$, we obtain for example the values

$$\zeta(12) = \frac{691\pi^{12}}{3^6 \times 5^3 \times 7^2 \times 11 \times 13}, \qquad (70)$$

$$\zeta(14) = \frac{2\pi^{14}}{3^6 \times 5^2 \times 7 \times 11 \times 13}. \qquad (71)$$

Finally, using our general formula for the case $p = 9$ and solving for $\sigma(9,\nu)$, once more with some help from the free-software program `maxima`, we obtain

$$\sigma(9,\nu) = \frac{P_9(\nu)}{Q_9(\nu)}, \tag{72}$$

where the two polynomials are given by

$$\begin{aligned} P_9(\nu) &= 715\,\nu^7 + 22287\,\nu^6 + 291104\,\nu^5 + 2066618\,\nu^4 + \\ &\quad +8671671\,\nu^3 + 21789479\,\nu^2 + 29485822\,\nu + 15144368, \qquad (73) \\ Q_9(\nu) &= 2^{17}(\nu+1)^9(\nu+2)^4(\nu+3)^3(\nu+4)^2 \times \\ &\quad \times(\nu+5)(\nu+6)(\nu+7)(\nu+8)^2(\nu+9). \qquad (74) \end{aligned}$$

Up to the case $p = 8$ it is possible and in fact fairly easy to verify the resulting formulas numerically with good precision, with the use of standard computational facilities. However, in this $p = 9$ case it is just too difficult to verify this formula by numerical means, except for $\nu = 0$, using the usual double-precision floating-point arithmetic. In order to do this one would have to use quadruple precision or better numerical arithmetic. The difficulty seems to lie in the direct numerical calculation of the sum $\sigma(9,\nu)$, not in the evaluation of the ratio of polynomials. Hence, the results discussed here acquire an algorithmic, numerical significance, enabling one to easily calculate the values of the sums.

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# 10 Historical Note

From the information supplied by Prof. Muldoon it seems that, since the time of Euler, aspects of the subject of these sums has been discovered and rediscovered, possibly more than just once. The explicit expressions of the sums as functions of $\nu$ were given, for the first 12 cases, by Lehmer [8]. The relation of the sums to the diffusion equation was first established by Kapitsa [9, 10], who also worked on the calculation of the sums. The recursive general solution presented here seems to be equivalent to a known recursion formula, first derived by Meiman [11] and later rediscovered by Kishore [12].

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