On the Convergence of a Certain Class of Fourier Series

Jorge L. deLyra
Department of Mathematical Physics
Physics Institute
University of São Paulo

August 12, 2010 – October 15, 2011

Abstract

A convergence criterion for real Fourier series, which is based solely on the behavior of the Fourier coefficients, is proposed and demonstrated in a simple case. The proof of convergence is based on a geometrical construction on the complex plane, and can be understood as an application of the Dirichlet convergence test. The class of convergent Fourier series thus defined is then extended to several other cases. The possible algorithmic applications of the result are pointed out, and the analytical character of the resulting set of limiting functions is briefly commented on.

1 Introduction

The work presented here resulted as a by-product of the development of a course of lectures on mathematical physics. In this course the Fourier series is used in its usual role for the solution of boundary value problems in partial differential equations. The convergence problem appears naturally in such a context, but leads to a point of view slightly different from that of the usual treatment. The usual criteria for convergence are based on the analytical properties of the function to which the series is associated. The known convergence theorems give sufficient conditions for the convergence of the series associated to a function. So far as the author knows, the formulation of a necessary and sufficient condition is still an open problem.

However, in applications to partial differential equations, what one has by the end of the day is the set of Fourier coefficients rather than the values of the function, so it is important to have convergence criteria based solely on the behavior of these coefficients, rather than on the analytical properties of the limiting function. This is the type of convergence criteria discussed in this paper. The criteria of this type for the absolute and uniform convergence of the series are well-known, and will be reviewed here in the context of a geometrical representation on the complex plane. We then focus on the criteria for the case in which there is simple point-wise convergence but not absolute or uniform convergence.

Let us review then some of the basic concepts, in order to establish context and notation. Let $f(\theta)$ be a real-valued function of the real variable θ, with domain in the interval $[-\pi, \pi]$, on which we impose periodic boundary conditions, that is, with $-\pi$ and π identified as the same point. No constraints other than that it be bounded and integrable are imposed on the function $f(\theta)$. The Fourier series associated with $f(\theta)$ is given by

$$S(\theta) = \frac{\alpha_0}{2} + \lim_{n\to\infty} \sum_{k=1}^{n} \left[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)\right],$$

where the coefficients are given by

$$\begin{aligned}\alpha_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, f(\theta)\cos(k\theta),\\ \beta_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, f(\theta)\sin(k\theta).\end{aligned}$$

We will indicate the series symbolically as

$$S(\theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\left[\alpha_k\cos(k\theta) + \beta_k\sin(k\theta)\right],$$

which is a notation indicating that there is an implicit limiting procedure involved. The set of functions

$$\begin{aligned}&1,\\ &\cos(k\theta), \quad \text{for} \quad k = 1,\ldots,\infty,\\ &\sin(k\theta), \quad \text{for} \quad k = 1,\ldots,\infty,\end{aligned}$$

is complete, which means that it can be used as a basis to represent all sufficiently well-behaved bounded periodic functions on the interval $[-\pi,\pi]$. This means that, if the series $S(\theta)$ converges at all, then it converges to the function $f(\theta)$, with the possible exception of a set of isolated points within the interval, which constitutes a zero-measure set. At points where $f(\theta)$ is continuous the series, if convergent, converges to the function. At points where it is discontinuous, the series, if convergent, converges to the average of the two lateral limits of $f(\theta)$ at that point, if these limits exist.

All these are well-known facts concerning the Fourier series [1]. In most of this paper we will not be concerned with the nature of the limit of the series, but only with the fact of its convergence. Since the set of $\cos(k\theta)$ functions and the set of $\sin(k\theta)$ functions are independent sets, the complete series can be convergent only if the separate series of $\cos(k\theta)$ functions and of $\sin(k\theta)$ functions converge independently. One can see this most simply by decomposing $f(\theta)$ in its even and odd parts,

$$f(\theta) = \frac{f(\theta)+f(-\theta)}{2} + \frac{f(\theta)-f(-\theta)}{2} = f_e(\theta) + f_\theta(\theta),$$

a decomposition that can always be made, regardless of any properties of the function. The same decomposition can be used for $S(\theta) = S_e(\theta) + S_o(\theta)$, and since the trigonometric functions involved have well-defined parities, this decomposition leads to

$$\begin{aligned}S_e(\theta) &= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\alpha_k\cos(k\theta),\\ S_o(\theta) &= \sum_{k=1}^{\infty}\beta_k\sin(k\theta),\end{aligned}$$

where the coefficients may be written in terms of the parts of the function $f(\theta)$ with the corresponding parities,

$$\alpha_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, f_e(\theta)\cos(k\theta),$$
$$\beta_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, f_o(\theta)\sin(k\theta).$$

We see therefore that the even and odd parts do not mix, and if one is to determine whether or not the Fourier series of a given function converges, one must do so independently for the even and odd parts of that function. In one case the series involved is purely a cosine series, and in the other purely a sine series. Each one of these series can be expressed as the real or imaginary part of a corresponding complex series, still with real coefficients,

$$\frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\alpha_k\cos(k\theta) = \frac{\alpha_0}{2} + \Re\left[\sum_{k=1}^{\infty}\alpha_k e^{\imath k\theta}\right],$$
$$\sum_{k=1}^{\infty}\beta_k\sin(k\theta) = \Im\left[\sum_{k=1}^{\infty}\beta_k e^{\imath k\theta}\right].$$

If we use the generic name a_k for the real coefficients α_k or β_k, as the case may be, we may, in a simpler way and with complete equivalence with the original problem, examine the convergence of the complex series with real coefficients a_k given by

$$S(\theta) = \sum_{k=0}^{\infty} a_k e^{\imath k\theta}.$$

The Fourier basis can therefore be represented, in its complex form, by the set of functions

$$e^{\imath k\theta}, \quad \text{for} \quad k = 0, \ldots, \infty.$$

It should be noted that this is *not* the complex version of the Fourier series, which would in general involve complex coefficients. It is simply a construction that allows us to study, in a simple and convenient way, the convergence of both the cosine series and the sine series, with the same arbitrary set of real coefficients a_k, in terms of the properties of these coefficients.

2 Geometrical Representation on the Complex Plane

Since the complex functions $\exp(\imath k\theta)$ can be represented as unit vectors in the complex plane $z = x + \imath y$, the partial sums of the series $S(\theta)$, which are given by

$$S_n(\theta) = \sum_{k=0}^{n} a_k e^{\imath k\theta},$$

can be represented very simply and elegantly on the complex plane, because the direction of each vector $a_k \exp(\imath k\theta)$ is given solely by the unit-length basis element $\exp(\imath k\theta)$, while its length is given solely by $|a_k|$. The partial sum $S_n(\theta)$ is a sum of n such vectors, which can be represented by drawing the vectors consecutively on the plane, starting from the origin, forming in this way a chain of complex vectors, formed by segments connected end-to-end at certain points, as illustrated in Figure 1. The complete series is represented by an infinite chain of such vectors. The set of vertex points along this infinite chain is identical to the set of partial sums of the series. Therefore, the convergence of the series is translated in

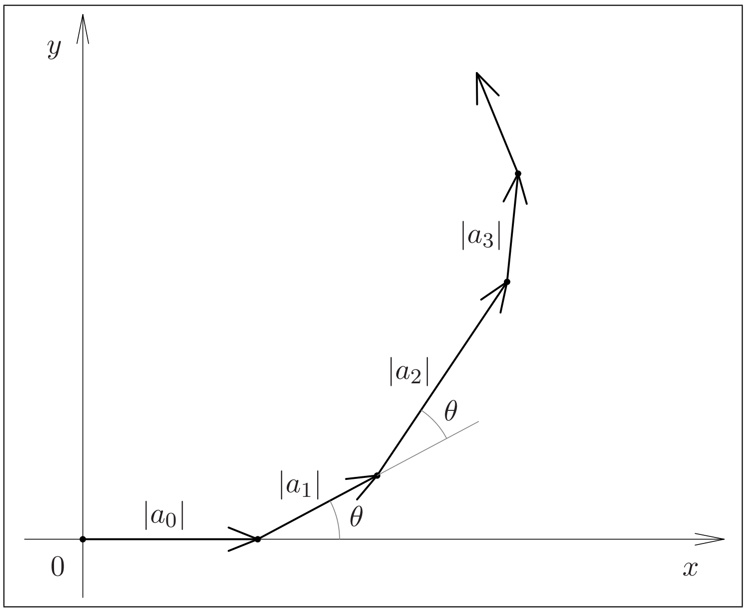

Figure 1: A generic chain of vectors $a_k \exp(\imath k\theta)$ on the complex plane $z = x + \imath y$.

this geometrical representation onto the fact that this sequence of points has a limit, and approaches some fixed point of the complex plane.

It now becomes immediately clear that the absolute convergence of the series is equivalent to the complete chain of vectors having a finite total length. One can see this because the series is absolutely convergent if and only if the sum of the moduli of the coefficients is finite, since the basis elements have unit moduli. In short, absolute converge means that the sum

$$\bar{S}_n = \sum_{k=0}^{n} |a_k|$$

converges to a finite limit when $n \to \infty$. Since the modulus of each coefficient gives the length of one of the vectors in the chain, this sum is the total length of the chain. If the series is absolutely convergent, then this sum is finite and so is the length of the chain. Likewise, if the chain has a finite length then this sum is finite and the series is absolutely convergent. If the series $S(\theta)$ is absolutely convergent, then it is also convergent, and hence both the cosine and the sine series, its real and imaginary parts, are convergent.

This relation is intuitively clear, since it is obvious that, if the chain has a finite total length, then it cannot extend to infinity, nor can it oscillate indefinitely between two points. It must extend a finite amount and hence stop at some point of the complex plane, which is its point of convergence. In this paper we will concern ourselves mostly with the case in which the series is convergent, but not absolutely convergent. In this case the chain has infinite total length, and then the series may fail to converge by extending to infinity or by oscillating indefinitely.

It is clear that the point $\theta = 0$ presents a particular convergence behavior, since for this value all the basis elements become equal to one, and hence the chain extends only over the real axis of the complex plane. It the series is not absolutely convergent, the chain has infinite length, and may easily extend to infinity. This is not certain only due to the possibility of successive exchanges of the signs of a_k, since the cancellations thus introduced may make the series converge, with the chain folding repeatedly over itself.

It is important to recall at this point that, if the series is not absolutely convergent, then the order in which the sum is executed may affect the final result. In this case, any statement of convergence must state the order in which the summation is meant to proceed. When we draw the chain in the complex plane, we are automatically adopting a particular order, the natural order of the sequence, with increasing values of the index k. This order will always be assumed here, unless otherwise noted.

Next, we will prove a theorem concerning the convergence of the complex series $S(\theta)$, that involves a criterion based on the behavior of the real coefficients a_k.

3 Proof of a Simple Convergence Theorem

Let us assume that the coefficients a_k of $S(\theta)$ are all positive and that they constitute a sequence which decreases to zero monotonically, that is, we assume the hypotheses

$$\begin{aligned} a_k &\geq 0, \forall k; \\ a_{k+1} &\leq a_k, \forall k; \\ \lim_{k\to\infty} a_k &= 0. \end{aligned}$$

Note however that no statement at all is made about the rate in which a_k approaches zero. This is enough to ensure that the series is convergent for $\theta \neq 0$, as can be established by the use of the Dirichlet convergence test. Nevertheless, we will give a proof based on the geometrical representation of the series on the complex plane, because it allows us to examine the behavior of the series in the case $\theta = 0$ and leads to an efficient summation technique in the convergent cases. We will show that these simple hypotheses suffice to ensure the convergence of the series $S(\theta)$ almost everywhere, in a sense which will be made clear during the argument. The result we will obtain here can be easily extended from this case to several others, but for definiteness let us consider only this case for the moment.

Interestingly, it is useful to analyze first a simple case that does not quite satisfy these hypotheses, namely the case in which $a_k = 1$ for all values of k. This case does not satisfy our hypotheses since this sequence of coefficients fails to approach zero when $k \to \infty$, which implies that the series must diverge. From now on we will denote the $k = 1$ basis element by v,

$$v = e^{\boldsymbol{\imath}\theta},$$

so that in this simple case the partial sums of the series $S(\theta)$ can be written as sums of powers of v,

$$S_n(\theta) = \sum_{k=0}^{n} e^{\boldsymbol{\imath}k\theta} = \sum_{k=0}^{n} v^k,$$

where v^k is a unit vector making an angle of value $k\theta$ with the real axis. In this simple case it is not difficult to verify, using the geometry of the complex plane, that the partial sums of the vectors v^k run indefinitely around a circle, making infinite turns around it, as illustrated in Figure 2. The series fails therefore to converge, and oscillates indefinitely instead. There is, of course, an exception in the case $\theta = 0$, since in this case all the vectors v^k are equal to one and therefore the series diverges linearly to positive infinity along the real axis. For any other value of θ the partial sums turn around a circle with a certain finite radius.

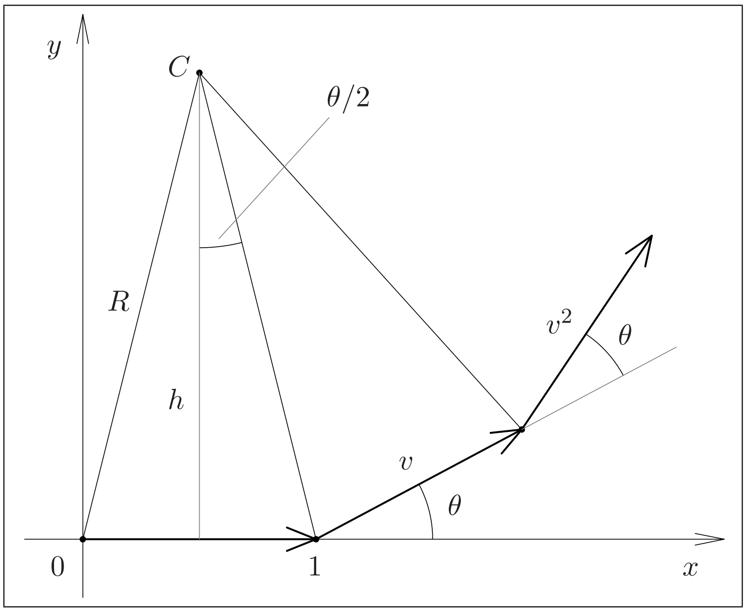

Figure 2: The chain of vectors corresponding to $a_k = 1$ for all k.

From the geometry shown in Figure 2 it is not difficult to verify that the radius R of the circle is given by

$$R = \frac{1}{2|\sin(\theta/2)|}.$$

The position of the center of rotation C is also easily verified to be given by

$$C = \left(\frac{1}{2}, h\right) = \left(\frac{1}{2}, \frac{1}{2\tan(\theta/2)}\right),$$

so that regardless of the value of θ the center is over the vertical line $x = 1/2$ of the complex plane $z = x + \imath y$. However, it is more interesting and useful to develop a different version of the formula for the position of C, based solely on complex arithmetic, which produces a simple result in terms of v. We will do this in steps.

First, consider the first vector $v^0 = 1$ alone, and how to go from its tip to the center C. One can do this by going back to the middle of the vector, and then proceeding perpendicularly to it by a length h. The first step is accomplished by subtracting $v^0/2$. For the second step, we consider the unit vector $\imath v^0$, which is perpendicular to v^0, being rotated from it by $\pi/2$ in the positive (counterclockwise) direction. In order to go to the center, we must then add $\imath v^0 h$, so that the complete path from the origin to C is given arithmetically by

$$\begin{aligned} C &= v^0 - \frac{1}{2}v^0 + \imath h v^0 \\ &= v^0 + \left(\imath h - \frac{1}{2}\right) v^0. \end{aligned}$$

It is easy to verify that this is the same formula for C given before. Similarly, we can get to the center from the tip of the second vector v^1, and the complex arithmetic representing the complete path in this case is

$$\begin{aligned} C &= v^0 + v^1 - \frac{1}{2}v^1 + \imath h v^1 \\ &= v^0 + v^1 + \left(\imath h - \frac{1}{2}\right) v^1. \end{aligned}$$

It is now easy to generalize this expression, representing arithmetically the path from the origin to the center, through the tip of the n-th vector of the chain, by

$$C = \sum_{k=0}^{n} v^k + \left(\imath h - \frac{1}{2}\right) v^n.$$

It is clear that, since the center of the circle is a fixed point in this simple case, all these formulas, for any value of n, produce the same result, and that the complex number C above does not really depend on n. The comparison of the first two formulas, for $n = 0$ and $n = 1$, allows us to write the factor $(\imath h - 1/2)$ in terms of v,

$$\begin{aligned} 1 + \left(\imath h - \frac{1}{2}\right) &= 1 + v + \left(\imath h - \frac{1}{2}\right) v \Rightarrow \\ \left(\imath h - \frac{1}{2}\right) &= \frac{v}{1-v}. \end{aligned}$$

Therefore, the position of the center is given by any one of the family of formulas

$$\begin{aligned} C &= \sum_{k=0}^{n} v^k + \frac{v}{1-v} v^n \\ &= \sum_{k=0}^{n} v^k + \frac{v^{n+1}}{1-v}. \end{aligned}$$

We see therefore that C can be written in terms of the partial sums S_n of the $S(\theta)$ series,

$$C = S_n + \frac{v^{n+1}}{1-v}.$$

This is true even for the case $n = 0$, which gives us the simplest form for C,

$$C = \frac{1}{1-v}.$$

We see here that, in the limit $\theta \to 0$, for which $v \to 1$, this point diverges to infinity, and corresponds to a circle with infinite radius that goes through the origin. This is the case in which the series diverges to infinity along the real axis, while C goes to infinity along the line $x = 1/2$. For any other value of θ the center is at a finite distance from the origin, and the series runs around it indefinitely.

Let us now verify what happens when we generalize this analysis to the case in which a_k decreases monotonically to zero. We may limit the discussion to the case in which $a_0 = 1$ without loss of generality, since it is always possible to put a_0 in evidence on the whole series and rename the remaining coefficients accordingly. If we consider the new sum and corresponding chain of vectors, each vector has the same direction as before, but their lengths now decrease from $a_0 = 1$. It is easy to see that in this case the chain tends to spiral inwards within the original circle, as illustrated in Figure 3. Besides, the center of rotation is no longer fixed, but drifts from step to step in the summation sequence.

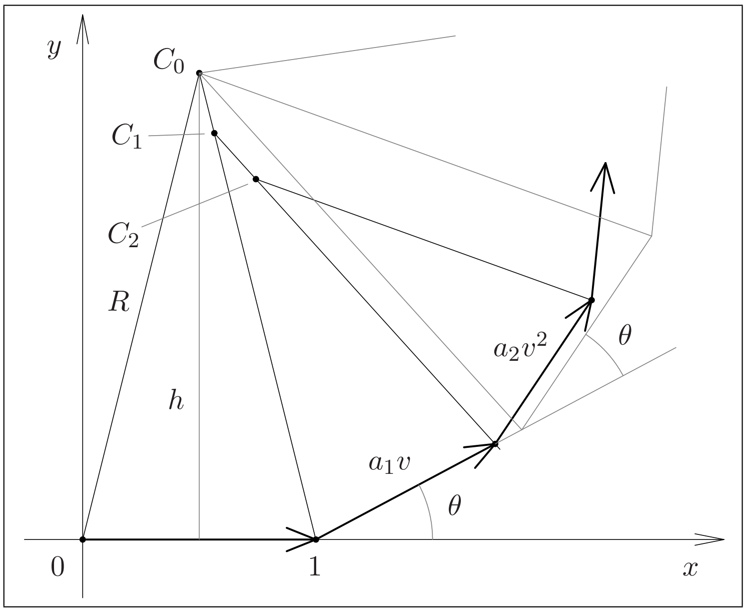

Figure 3: Example of a chain of vectors spiraling inward within the circle.

One can see that, if the coefficients a_k approach zero very slowly, then the center also drifts very slowly, while the chain of partial sums spirals around many times. On the other hand, if the coefficients a_k approach zero very fast, then the center drifts faster, and the chain of partial sums spirals around only a few times, if at all. The essential element of the proof of convergence is the idea of following the drift of the center, rather than the spiraling of the chain of partial sums.

We have therefore a sequence of points which we may call "instantaneous centers of rotation", $C_0, C_1, C_2, C_3, \ldots$, that corresponds to the sequence of partial sums $S_0, S_1, S_2, S_3, \ldots$. The same arithmetic arguments we used before can be used in this case to give C_n in terms of S_n. We now have the partial sums

$$S_n = \sum_{k=0}^{n} a_k v^k,$$

and the path to get to the center C_n via the tip of the chain S_n is represented arithmetically by

$$\begin{aligned} C_n &= S_n - \frac{1}{2} a_n v^n + \imath h a_n v^n \\ &= S_n + \left(\imath h - \frac{1}{2} \right) a_n v^n, \end{aligned}$$

where h is the same as before, leading to the expression, in terms of v only,

$$\begin{aligned} C_n &= S_n + \frac{v}{1-v} a_n v^n \\ &= S_n + \frac{a_n v^{n+1}}{1-v}. \end{aligned}$$

It is important to observe now that the distance from a partial sum to the corresponding center, which is given by $|C_n - S_n|$, is proportional to a_n,

$$
\begin{aligned}
|C_n - S_n| &= \left|\imath h - \frac{1}{2}\right| |a_n||v^n| \\
&= a_n\sqrt{h^2 + 1/4} \\
&= \frac{a_n}{2|\sin(\theta/2)|},
\end{aligned}
$$

and therefore goes to zero when $a_n \to 0$, so long as θ is not zero or a multiple of 2π. Therefore, the partial sums approach the drifting center in the $n \to \infty$ limit, and hence the two sequences, C_n and S_n, have the same limit.

We may now consider the series $C(\theta)$ corresponding to the partial sums C_n, and the corresponding chain of complex vectors, in analogy with what we did for the $S(\theta)$ series. We will now show that, if the coefficients a_n satisfy our hypotheses, then this series is in fact absolutely convergent, so long as $v \neq 1$. From the relation between C_n and S_n, multiplying by $(1 - v)$, we get

$$
\begin{aligned}
(1 - v)C_n &= (1 - v)S_n + a_n v^{n+1} \\
&= (1 - v)\sum_{k=0}^{n} a_k v^k + a_n v^{n+1}.
\end{aligned}
$$

In order to write the resulting expression as a power series in v, me manipulate the sums and indices in order to get

$$
\begin{aligned}
(1 - v)C_n &= \sum_{k=0}^{n} a_k v^k - \sum_{k=0}^{n} a_k v^{k+1} + a_n v^{n+1} \\
&= a_0 + \sum_{k=1}^{n} (a_k - a_{k-1}) v^k.
\end{aligned}
$$

If we take absolute values and use the triangular inequalities, we get

$$
|1 - v||C_n| \leq |a_0| + \sum_{k=1}^{n} |a_k - a_{k-1}|,
$$

since v has unit modulus. Since the coefficients a_k decrease monotonically to zero, we may write this as

$$
|1 - v||C_n| \leq a_0 + \sum_{k=1}^{n} (a_{k-1} - a_k).
$$

We now observe that in the remaining sum almost all terms cancel out in pairs, except for one a_0 and one a_n, so that we have

$$
|1 - v||C_n| \leq 2a_0 - a_n.
$$

If we now take the $n \to \infty$ limit, by our hypotheses a_n vanishes, and we get

$$
|1 - v||C(\theta)| \leq 2a_0.
$$

Therefore, since a_0 is a given finite real number, the series $(1 - v)C(\theta)$ is absolutely convergent and, so long as $v \neq 1$, so is the series $C(\theta)$, for which we have

$$|C(\theta)| \leq \frac{2a_0}{|1-v|}.$$

In either case the length of the chain associated to the series is finite, thus characterizing a situation of absolute convergence. This establishes the convergence of $C(\theta)$ for the case $\theta \neq 0$, and since S_n converges to C_n, it also establishes the convergence of $S(\theta)$.

Before we give a more rigorous proof of this result, let us discuss it in terms of the two separate real series, the one with the cosines and the one with the sines. If $\theta \neq 0$, then it is clear that both series converge. The single point $\theta = 0$ requires special consideration. In this case we have $v = 1$, the radius of the circle becomes infinite, and the complex series diverges to infinity along the real axis. This implies that only its real part diverges, while its imaginary part is in fact identically zero. Therefore, the conclusion is that the series of sines always converges, while the series of cosines converges at all points except one, the point $\theta = 0$.

We may use the geometrical ideas presented here in order to understand the origin of the discontinuity of the sine series at the special point $\theta = 0$. This discontinuity will appear when a_k decreases slowly with k, so that the series is not absolutely convergent, and the center of rotation drifts very little from its initial position. For θ small and positive the center C will be in the upper half-plane, far from the real axis, and the chain of vectors will curve slowly upwards, away from the real axis, in order to eventually converge to that center. However, if θ is small and negative, then the center C will be in the lower half-plane, again far from the real axis, and the chain of vectors will curve slowly downwards. We see therefore that there will be a discontinuity, since an infinitesimal variation that flips the sign of θ will result in a large jump in the imaginary part of the point of convergence, from strictly positive to strictly negative values.

4 Rigorous Proof of the Theorem

The same complex arithmetic used in the previous section can be employed for the construction a rigorous and direct demonstration that the sequence S_n is convergent. The argument is very similar to a demonstration of the Dirichlet convergence test. In order to do this we show below that, for $v \neq 1$, the sequence S_n is a Cauchy sequence within a closed disc in the complex plane, which implies that it is convergent. The demonstration is quite simple and short. We simply consider the partial sums

$$S_n = \sum_{k=0}^{n} a_k v^k,$$

multiply by $(1-v)$ and manipulate the summation indices in order to obtain, in a way similar to the manipulations done before,

$$(1-v)S_n = a_0 + \sum_{k=1}^{n} (a_k - a_{k-1})v^k - a_n v^{n+1}.$$

Taking now absolute values and using the triangular inequalities we get

$$|1-v||S_n| \leq |a_0| + \sum_{k=1}^{n} |a_k - a_{k-1}| + |a_n|.$$

Assuming now that the sequence of coefficients a_k decreases monotonically to zero, we may write

$$|1-v||S_n| \leq a_0 + \sum_{k=1}^{n}(a_{k-1}-a_k) + a_n.$$

Once again almost all terms in the remaining sum cancel out in pairs, and we are left with

$$|1-v||S_n| \leq 2a_0.$$

It follows that, so long as $v \neq 1$, we may write a finite upper bound to the partial sum,

$$|S_n| \leq \frac{2a_0}{|1-v|} = \rho,$$

which does not depend on n, and is therefore valid for all n. We conclude therefore that the whole set of partial sums is contained within a closed disc of radius ρ, centered at the origin, which is the initial point of the sequence. As a consequence, given any two partial sums S_n and $S_{n'}$, the distance between them is less than 2ρ,

$$|S_n - S_{n'}| \leq 2\rho,$$

for any n and any n'. It is not difficult to repeat this argument for a sum that starts at an arbitrary intermediate point m of the sequence, and goes up to a point $n > m$. This is the difference of two partial sums of the series,

$$\begin{aligned} S_{mn} &= S_n - S_m \\ &= \sum_{k=m+1}^{n} a_k v^k. \end{aligned}$$

Repeating for S_{mn} the same manipulations executed before for S_n, we get this time

$$(1-v)S_{mn} = a_{m+1}v^{m+1} + \sum_{k=m+2}^{n}(a_k - a_{k-1})v^k - a_n v^{n+1}.$$

Taking now absolute values we get

$$|1-v||S_{mn}| \leq 2a_{m+1}.$$

In this way we see that, so long as $v \neq 1$, all the elements of the sequence S_n with $n > m$ are within a closed disc centered at S_m, with a finite radius ρ_{m+1},

$$|S_{mn}| = |S_n - S_m| \leq \frac{2a_{m+1}}{|1-v|} = \rho_{m+1},$$

which is independent of n, so that this relation is valid for all n. It follows that any two elements S_n and $S_{n'}$ of the sequence such that $n > m$ and $n' > m$ are within this disc, and hence the distance between them is bounded by the diameter of the disc,

$$|S_{n'} - S_n| \leq 2\rho_{m+1}.$$

This establishes the Cauchy-sequence structure of our sequence S_n. In order to complete the argument, let a real number $\epsilon > 0$ be given. We consider the infinite collection of positive real numbers

$$2\rho_{m+1} = \frac{4a_{m+1}}{|1-v|},$$

all of which are finite so long as $v \neq 1$. Since we have by our hypotheses that $a_{m+1} \to 0$ when $m \to \infty$, it follows that there is a value of m such that

$$2\rho_{m+1} = \frac{4a_{m+1}}{|1-v|} < \epsilon.$$

If we consider this value of m, then it follows that, for any $n > m$ and any $n' > m$, it is true that

$$|S_{n'} - S_n| \leq 2\rho_{m+1} = \frac{4a_{m+1}}{|1-v|} < \epsilon,$$

which establishes that there is a value of m that satisfies the criterion for a Cauchy sequence. Since S_n is thus shown to be a Cauchy sequence within a closed disc, which is a complete set, it follows that it converges. As a consequence, so long as $v \neq 1$ the Fourier cosine and sine series that correspond to the sequence S_n are both convergent. The same comments made before about the special case $v = 1$ still apply, of course, so that the sine series converges everywhere, while the cosine series converges at all points except $\theta = 0$.

5 Extension of the Result to Other Cases

The result demonstrated here, which we may refer to as the monotonicity convergence criterion, can be readily extended to some other cases, which are easily related to the one we examined in detail above. First of all, the condition that the coefficients a_n decrease monotonically to zero need not apply to the whole sequence. It is enough if it holds for all n above a certain minimum value n_0, because the sum up to this value n_0 is a finite sum and hence there are no convergence issues for this initial part of the complete sum. Also, it is obvious that we may exchange the overall sign of the series and still have the result hold true, so that the coefficients may as well be monotonically increasing to zero from negative values.

Some other extensions are a bit less obvious. A particularly interesting one is that in which all coefficients with a certain parity of k are zero, while the others approach zero monotonically. For example, if the coefficients are only non-zero for $k = 2j$, then the partial sums are given by

$$S_n(\theta) = \sum_{j=0}^{n} a_{2j} e^{\imath 2j\theta}.$$

If we consider the angle $\theta' = 2\theta$ and rename the coefficients as $a'_j = a_{2j}$, we may write this as

$$S_n(\theta') = \sum_{j=0}^{n} a'_j e^{\imath j\theta'},$$

which has exactly the same structure as the case examined before, with only trivial differences in the names of the symbols. If we adopt $v = \exp(\imath\theta')$, we may write

$$S_n(\theta') = \sum_{k=0}^{n} a'_k v^k.$$

Therefore, so long as the coefficients a'_k approach zero monotonically when $k \to \infty$, the result holds for this type of series as well. The only significant difference is that, since θ' has as its domain the periodic interval $[-2\pi, 2\pi]$, there are now two special points, where the exceptional case $v = 1$ applies, $\theta' = 0$ and $\theta' = 2\pi$, corresponding to $\theta = 0$ and $\theta = \pi$. At these points the sine series will converge to a discontinuous function, while the cosine series will diverge, when the complex series is not absolutely convergent. On can easily see this because at $\theta = 0$ and $\theta = \pi$ we have $\sin(k\theta) = 0$ for all k, and therefore the sine series converges to zero, while at at $\theta = 0$ we have $\cos(k\theta) = 1$ for all k, and at $\theta = \pi$ we have $\cos(k\theta) = 1$ for all terms of the series because k is even, so that the cosine series diverges to infinity when the complex series is not absolutely convergent.

The complementary case can be treated as well. If the coefficients are only non-zero for $k = 2j + 1$, then the partial sums are given by

$$S_n(\theta) = \sum_{j=0}^{n} a_{2j+1} e^{\boldsymbol{\imath}(2j+1)\theta} = e^{\boldsymbol{\imath}\theta} \sum_{j=0}^{n} a_{2j+1} e^{\boldsymbol{\imath}2j\theta}.$$

Using once again the variable $\theta' = 2\theta$ and the coefficients $a'_j = a_{2j+1}$, we may write this as

$$S_n(\theta') = e^{\boldsymbol{\imath}\theta'/2} \sum_{k=0}^{n} a'_k e^{\boldsymbol{\imath}k\theta'},$$

so that once more the problem can be reduced to the previous one, and so the result holds in this case as well. Just as in the previous case, there are two special points in this one, $\theta' = 0$ and $\theta' = 2\pi$, corresponding to $\theta = 0$ and $\theta = \pi$. At these points the sine series converges to zero for the same reason as before, while at $\theta = 0$ we have $\cos(k\theta) = 1$ for all k, and at $\theta = \pi$ we have $\cos(k\theta) = -1$ for all terms of the series because k is odd, so that the cosine series diverges to positive or negative infinity when the complex series is not absolutely convergent. This case includes the paradigmatic example of the square wave, that has a series of sines with this type of structure, and two points of discontinuity.

It is immediately apparent that we may further extend the results to series with coefficients having alternating signs, given for example by $(-1)^k$, so long as both the positive and negative coefficients approach zero monotonically. This type of series can be separated as the sum of two sub-series, either one of which is convergent by our theorem, and therefore they converge as well, possibly with the exception of a couple of special points in the case of the cosine series. In fact, this can be generalized to cases where the original series can be decomposed into a finite number of disjoint sub-series, each of which has coefficients that approach zero monotonically, but in order to do this we must first discuss series with non-zero terms only every so many terms, instead of every other term.

This type of extension can be generalized without difficulty to sparser series, with non-zero coefficients only every so many terms, so long as they come with a regular step. The extensions just discussed correspond to the case of step 2, and in general the result holds as well for series with step p, which have then p special points. Also, in this case there are p alternatives for the position of the first element of the sequence of non-zero terms, which may lead to varying scenarios with regard to the convergence of the sine and cosine series at the special points. When there is no absolute convergence at least one of the two series, the sine series or the cosine series, will diverge at the special points. Any series which does converge will do so to a function which is discontinuous at these points. Except for these special points, any sparse series with a constant step and coefficients that approach zero monotonically for large values of k is convergent in almost all the periodic interval.

The identification of the special points of a given series can always be accomplished, by the use of the algorithm that follows. The divergence of the complex series at these points is always caused by the alignment of all the vectors v^k in the complex plane. This happens whenever the argument of the periodic functions returns to the same value from one element of the sequence of terms to the next, that is, when k varies by the length one step of the series, $\Delta k = p$. Therefore the special points are determined by the condition

$$\begin{aligned} \Delta k\theta &= n\,2\pi \Rightarrow \\ \theta &= \frac{n\,2\pi}{p}, \end{aligned}$$

for some integer n. The values of θ at the special points are determined by the values of n such that this angle falls within the periodic interval $[-\pi, \pi]$. One can verify that there are always p such values of n. One can then determine whether the sine and cosine series converge or diverge by substituting these values in these periodic functions. The series can converge when there is no absolute convergence only if the periodic function is zero for all values of k. Otherwise the series diverges to infinity. By using this simple algorithm one can always determine the state of convergence of each series in each special point.

We may further extend our result to what we may call step-κ monotonic series. These are series in which all the coefficients may be non-zero, but that can be separated into κ sub-series, each one with a uniform step of size κ. These step-κ monotonic series are those with coefficients which satisfy the condition

$$\lim_{k\to\infty} a_k = 0,$$

and the condition that there exist an integer κ such that

$$a_{k+\kappa} \leq a_k, \; \forall k,$$

above a certain minimum value of k. Since the number of component sub-series is finite, so long as all κ sub-series are convergent by the monotonicity criterion, it follows that their sum will necessarily converge as well. This establishes the convergence of the original series in all the domain except for a finite set of κ special points, which are common to all the component sub-series, and for which we cannot conclude anything by this method.

Observe that we may have both positive and negative coefficients coexisting in such a series, so long as a_k and $a_{k+\kappa}$ always have the same sign. The alternating-sign series discussed before, with coefficients with one sign for the even indices and the other sign for the odd indices, are simple examples of step-2 monotonic series.

A more general extension is simple to state, but is such that the series belonging to it are not so easy to identify. One may consider the set of all finite linear combinations of component series which have coefficients that approach zero monotonically. The component series may all have steps different from each other, and their coefficients may approach zero at different and arbitrary rates. So long as the number of series in the linear combination is finite, the resulting series is convergent, possibly with the exception of a finite set of special points. Note that in this case the resulting series may not have coefficients that approach zero monotonically at all.

At last, since the series $S(\theta)$ and $C(\theta)$ converge to the same point, so long as the coefficients a_k tend to zero for $k \to \infty$, and since the series $C(\theta)$ converges typically much faster than $S(\theta)$, in any case in which the series $C(\theta)$ converges at all, even if does not do so absolutely, the series $S(\theta)$ must converge as well. Next we elaborate on the algorithmic significance of the relation between these two series.

6 Possible Algorithmic Applications

The complex arithmetic used in the demonstration of the theorem can be used, not only to prove the convergence of the series, but also as an algorithm to accomplish an efficient numerical evaluation of the limit of the series. This can be very useful, because series $S(\theta)$ of the type examined here may converge extremely slowly. Let us recall that we have for the partial sums of the series $(1-v)S(\theta)$

$$(1-v)S_n = a_0 + \sum_{k=1}^{n}(a_k - a_{k-1})v^k - a_n v^{n+1}.$$

By our hypotheses, in the $n \to \infty$ limit the last term vanishes, and therefore we may write for the series

$$(1-v)S(\theta) = a_0 + \sum_{k=1}^{\infty}(a_k - a_{k-1})v^k.$$

Observe that all the terms in the sum in the right-hand side are negative, due to the fact that the coefficients decrease monotonically. Unlike the original series for $S(\theta)$, the series in the right-hand side in this formula is always absolutely convergent, as we saw during the proof of the theorem. This series converges therefore much faster than the original one, when that original one is not absolutely convergent. We may define new symbols b_k for its coefficients,

$$b_k = a_k - a_{k-1}, \text{ for } k = 1, \ldots, \infty,$$

while for $k = 0$ we adopt the definition $b_0 = a_0$, in terms of which we may write that

$$S(\theta) = \frac{1}{1-v}\sum_{k=0}^{\infty} b_k v^k,$$

which holds so long as $v \neq 1$. In order to identify the real and imaginary parts of this expression, which are related respectively to the series of cosines and the series of sines, we write it back in terms of θ, obtaining

$$S(\theta) = \frac{1}{1-e^{\imath\theta}}\sum_{k=0}^{\infty} b_k e^{\imath k\theta}.$$

The first factor in the right-hand side can be written as

$$\frac{1}{1-e^{\imath\theta}} = \frac{\imath}{2}\,\frac{e^{-\imath\theta/2}}{\sin(\theta/2)}.$$

If we substitute this in the expression for the series, we get

$$S(\theta) = \frac{\imath}{2\sin(\theta/2)} \sum_{k=0}^{\infty} b_k e^{\imath(k\theta - \theta/2)}.$$

From this we may easily identify the series of sines and of cosines, and in this way we obtain for them the relations

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \cos(k\theta) &= \frac{-1}{2\sin(\theta/2)} \sum_{k=0}^{\infty} b_k \sin(k\theta - \theta/2), \\
\sum_{k=1}^{\infty} a_k \sin(k\theta) &= \frac{1}{2\sin(\theta/2)} \sum_{k=0}^{\infty} b_k \cos(k\theta - \theta/2).
\end{aligned}$$

Separating once again the $k = 0$ terms, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \cos(k\theta) &= \frac{a_0}{2} - \frac{1}{2\sin(\theta/2)} \sum_{k=1}^{\infty} b_k \sin[(k - 1/2)\theta], \\
\sum_{k=1}^{\infty} a_k \sin(k\theta) &= \frac{a_0}{2} \frac{\cos(\theta/2)}{\sin(\theta/2)} + \frac{1}{2\sin(\theta/2)} \sum_{k=1}^{\infty} b_k \cos[(k - 1/2)\theta].
\end{aligned}$$

The initial terms of these two series can be recognized as the coordinates $a_0/2$ and $a_0 h$ of the initial center of rotation. Note that the other terms in the right-hand sides have half-integers in their arguments, rather than the usual integers of the Fourier series. They may be interpreted as well as series over only odd indices $(2k - 1)$, with the half-angle $\theta/2$ as argument.

For coefficients a_k that satisfy the hypotheses of our theorem and that approach zero too slowly for the original series to be absolutely convergent, the series in the right-hand sides converge much faster than the corresponding ones in the left-hand sides, and therefore can be used to calculate approximations to the same limits in a much more efficient way when $v \neq 1$. This constitutes therefore a numerical technique for the calculations of these limits. By using this technique one approaches the limit by following the drift of the center $C(\theta)$, rather than the propagation of the chain of the original series $S(\theta)$. In general, even if the coefficients a_k do not satisfy our hypothesis of monotonicity, whenever the series

$$\sum_{k=0}^{\infty} b_k v^k$$

converges, so does the original series $S(\theta)$, so long as a_k tends to zero as $k \to \infty$. Hence, this numerical technique may be useful even in cases when not all our hypotheses are satisfied.

Each one of the extensions of the theorem to other cases, which were discusses previously, also comes with corresponding summation formulas similar to the ones presented above, which in each case may be used for the efficient numerical estimation of the limits. For example, we saw that the result can be extended to series with non-zero coefficients only for even k, that is for $k = 2j$, and if we consider the angle $\theta' = 2\theta$ we may write such a series as

$$S_n(\theta') = \sum_{j=0}^{n} a_j e^{\imath j\theta'},$$

which has exactly the same structure as the series just discussed, so that we may write at once that, for $\theta' \neq 2n\pi$ for all integers n,

$$\begin{aligned}\sum_{j=0}^{\infty} a_j \cos(j\theta') &= \frac{a_0}{2} - \frac{1}{2\sin(\theta'/2)} \sum_{j=1}^{\infty} b_j \sin[(j-1/2)\theta'],\\ \sum_{j=1}^{\infty} a_j \sin(j\theta') &= \frac{a_0}{2}\,\frac{\cos(\theta'/2)}{\sin(\theta'/2)} + \frac{1}{2\sin(\theta'/2)} \sum_{j=1}^{\infty} b_j \cos[(j-1/2)\theta'],\end{aligned}$$

where $b_j = a_j - a_{j-1}$, for $j = 1, \ldots, \infty$, and where once more the two initial terms are the coordinates $a_0/2$ and $a_0 h$ of the initial center of rotation. Writing directly in terms of θ we have

$$\begin{aligned}\sum_{j=0}^{\infty} a_j \cos(2j\theta) &= \frac{a_0}{2} - \frac{1}{2\sin(\theta)} \sum_{j=1}^{\infty} b_j \sin[(2j-1)\theta],\\ \sum_{j=1}^{\infty} a_j \sin(2j\theta) &= \frac{a_0}{2}\,\frac{\cos(\theta)}{\sin(\theta)} + \frac{1}{2\sin(\theta)} \sum_{j=1}^{\infty} b_j \cos[(2j-1)\theta],\end{aligned}$$

where we have now the condition that $\theta \neq n\pi$ for all integers n. Note that in this case, while the series on the left are Fourier series over only the even indices, the ones on the right are over only odd indices. The case in which the series has non-zero coefficients only for odd k, that is for $k = 2j+1$, can be treated in a similar way. As we saw before, in this case we may write the series as

$$e^{-\imath\theta'/2} S_n(\theta') = \sum_{j=0}^{n} a_j e^{\imath j\theta'},$$

which once more has the same structure as before, except for the additional overall exponential factor. We get therefore in this case, for the series written in terms of b_j,

$$\begin{aligned}e^{-\imath\theta'/2} S(\theta') &= e^{-\imath\theta'/2} \frac{\imath}{2\sin(\theta'/2)} \sum_{j=0}^{\infty} b_j e^{\imath j\theta} \Rightarrow\\ S(\theta') &= \frac{\imath}{2\sin(\theta'/2)} \sum_{j=0}^{\infty} b_j e^{\imath j\theta},\end{aligned}$$

which then leads to

$$\begin{aligned}\sum_{j=0}^{\infty} a_j \cos[(j+1/2)\theta'] &= 0 - \frac{1}{2\sin(\theta'/2)} \sum_{j=1}^{\infty} b_j \sin(j\theta'),\\ \sum_{j=0}^{\infty} a_j \sin[(j+1/2)\theta'] &= \frac{a_0}{2\sin(\theta'/2)} + \frac{1}{2\sin(\theta'/2)} \sum_{j=1}^{\infty} b_j \cos(j\theta'),\end{aligned}$$

where $b_j = a_j - a_{j-1}$, for $j = 1, \ldots, \infty$, and where we see that the two coordinates of the initial center of rotation are now 0 and $a_0 R$. Writing directly in terms of θ we have

$$\begin{aligned}\sum_{j=0}^{\infty} a_j \cos[(2j+1)\theta] &= -\frac{1}{2\sin(\theta)} \sum_{j=1}^{\infty} b_j \sin(2j\theta),\\ \sum_{j=0}^{\infty} a_j \sin[(2j+1)\theta] &= \frac{a_0}{2\sin(\theta)} + \frac{1}{2\sin(\theta)} \sum_{j=1}^{\infty} b_j \cos(2j\theta),\end{aligned}$$

where we have the same condition that $\theta \neq n\pi$ for all integers n. Note that in this case, while the series on the left are Fourier series over only the odd indices, the ones on the right are over only even indices.

7 Analytical Character of the Limits

It is interesting that one may establish without too much difficulty at least one analytical property of the functions to which the class of Fourier series studied here converge, which is related to their continuity over the unit circle. For simplicity, let us limit this discussion to the simplest type of series, the one which we discussed in detail in this paper. However, there should be no difficulty in extending the ideas to the generalizations mentioned before.

First let us observe that, for series of bounded functions such as the complex basis $v^k = \exp(\imath k\theta)$, for which the modulus of each function is unity, and hence independent of the variable θ, absolute convergence implies convergence with criteria which do not depend on the variable θ, and therefore imply uniform convergence. This can also be established by the use of the Weierstrass M-test. Hence, given any closed interval of the unit circle which does not contain the special point $\theta = 0$, for which $v = 1$, the series in the following expression for $S(\theta)$,

$$S(\theta) = \frac{1}{1-v}\sum_{k=0}^{\infty} b_k v^k,$$

is uniformly convergent within that interval. Therefore, using the well-known fact that a series of continuous functions which converges uniformly over a closed interval does so to a continuous function, it follows that the series converges to a continuous function within that interval. This implies that the whole class of Fourier series discussed here, in this simplest case, converge to continuous functions, everywhere except for the special point $\theta = 0$ over the unit circle. It is clear that similar properties hold for the extensions of the theorem in which there are several such special points.

However, the differentiability properties of these functions are not so easily determined. Since the series in the right-hand side is absolutely and hence uniformly convergent, it may be possible to differentiate it term-by-term to produce a convergent series, even if one certainly cannot do the same to the original series for $S(\theta)$, when that series is not absolutely convergent. Using the fact that

$$\frac{\partial v}{\partial \theta} = \imath v,$$

we may attempt to calculate the derivative of $S(\theta)$, thus obtaining

$$\begin{aligned} \frac{\partial S}{\partial \theta} &= \frac{\imath v}{(1-v)^2}\sum_{k=0}^{\infty} b_k v^k + \frac{\imath v}{1-v}\sum_{k=1}^{\infty} k\, b_k v^{k-1} \\ &= \frac{\imath v}{1-v}\, S(\theta) + \frac{\imath}{1-v}\sum_{k=1}^{\infty} k\, b_k v^k. \end{aligned}$$

The series contained within $S(\theta)$ in the first term is convergent due to our theorem, so the differentiability of the function is related to the convergence or lack of convergence of the series in the second term, which depends on the behavior of the coefficients $c_k = k\, b_k$. If these coefficients turn out to be such that the corresponding series is absolutely convergent within the closed interval mentioned before, then it follows that the function is differentiable in that interval.

Absolute convergence depends on the coefficients c_k going to zero sufficiently fast with k, while term-by-term differentiation implies in a factor of k multiplying b_k. Therefore, while it is possible that some of these functions are continuous, it seems likely that many are not. In fact, since the original series defining $S(\theta)$ may converge very slowly and may contain significant Fourier components for very large frequencies, it does lead to the impression that the limiting functions might not be differentiable at all, undergoing violent high-frequency oscillations everywhere. At first glance it certainly seems very unlikely that these functions may have more than the first derivative, due to the increasing powers of k multiplying b_k, which are implied by multiple term-by-term differentiation.

However, there is another possibility, namely that the coefficients c_k turn out to also satisfy the hypotheses of our theorem. In this case one would be able to transform the corresponding series once more into one which is absolutely convergent, which would then imply the differentiability of the function. For cases in which this procedure can be iterated, it might be possible to show that the functions also have the second derivative, or even higher-order ones. The crucial hypothesis is that of the monotonicity of c_k, and that depends essentially on the monotonicity of b_k. If there are series for which the transformation from a_k to b_k preserves the monotonic character of the sequence, then these series should converge to functions which are differentiable in the closed interval mentioned above.

Conceivably, there might even exist series for which this procedure of the transformation of the coefficients can be iterated indefinitely. In such a case the limiting functions would turn out to be infinitely differentiable within the closed interval, that is, of class $\mathcal{C}^{\infty}$. We must conclude, therefore, that the differentiability properties of these functions remain as completely open questions.

8 Final Comments

The result demonstrated here may constitute an useful tool for those who need to use Fourier series in their work, particularly in the solution of boundary value problems in the theory and the applications of partial differential equations. However, at least two interesting questions remain as open problems. One is of practical interest, and regards the issue of how to recognize, in general, when a given series can be decomposed as a finite sum of series with coefficients which satisfy our hypotheses.

Another question, of a more theoretical interest, is whether or not one can cover all possible convergent Fourier series with some form of decomposition in series such as those we studied here. In other words, whether or not there is a convergent Fourier series which is not absolutely convergent and cannot be so decomposed. This may constitute a new path of exploration for the old quest for a necessary and sufficient condition for the convergence of Fourier series to limiting functions.

It also is interesting that the theorem gives rise to the possibility of some numerical and possibly also symbolical experimentation with the construction of functions, via the evaluation of the limits of series which converge extremely slowly. Using the series with the b_k coefficients, the limits can be easily evaluated, but the corresponding $S(\theta)$ functions, as defined by the series with the a_k coefficients, will have significant components for very large frequencies. For example, one could consider further probing into the analytical nature of the limit of the series

$$S(\theta) = \sum_{k=1}^{\infty} \frac{v^k}{k^p},$$

for small positive values of p, typically $p < 1$, but preferably much smaller positive numbers. In particular, it would be interesting to determine whether or not there are any weakly-convergent Fourier series of this type which converge to $\mathcal{C}^\infty$ functions.

Finally, due to the question of the order of summation of series which are not absolutely convergent, and to the fact that a particular order was implicitly chosen in some of the extensions of the theorem which were discussed before, it may be interesting to examine more carefully these extensions, possibly aiming at the formulation of more precise proofs when appropriate.

References

[1] See, for example, R. V. Churchill, "Fourier Series and Boundary Value Problems", McGraw-Hill Book Co., New York, 1963, or R. V. Churchill and J. W. Brown, "Fourier Series and Boundary Value Problems", McGraw-Hill Book Co., Singapore, 1987, and the references therein.