Numerical Tests of Center Series

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Abstract

We present numerical tests of several simple examples of center series, with the aim of evaluating the speed with which they converge, as compared to the corresponding Fourier series. These center series were defined and developed in previous papers, and constitute an improved form of trigonometric expansion of real functions, related to the Fourier series. The tests performed are comparative ones, between each Fourier series and the corresponding first-order center series. They show that, specially in the case of Fourier series that converge very slowly, the use of the center series can represent a truly enormous numerical and computational advantage. We also use the numerical advantage provided by the center series to display the Fourier Conjugate functions of each example worked out, as well as to test the speed of convergence of their Fourier series and first-order center series. For the execution of the comparative tests between the Fourier and first-order center series it was necessary, in some cases, to use also the second-order center series, as tools for the tests, since these converge even faster than the first-order center series.

1 Introduction

In a previous paper [2] we introduced the concept of the center series. This constitutes an improved form of trigonometric expansion of real functions. Starting from the Fourier expansion of a given real function one may construct from it, even in cases where the Fourier series is outright divergent, other expressions that converge to the function that originated the Fourier coefficients, involving certain trigonometric series which we name "center series", and which have better convergence characteristics than the original Fourier series. Although in [2] we give explicit examples only of first-order center series, as commented on that paper it is also possible to construct higher-order center series, with even better convergence characteristics. In fact, in this paper we will have the opportunity to calculate and use some second-order center series.

The construction of center series from Fourier series can be understood and executed in the context of a correspondence, which was established in an earlier paper [1], between Definite Parity (DP) Fourier series and certain analytic functions $w(z)$ on the unit disk of the complex plane, which we refer to as "inner analytic functions". In this context the construction of center series is associated to an operation of factorization of the singularities of $w(z)$ in the complex plane, as explained in [2]. Since center series are currently rather unfamiliar objects, we present in an appendix of this paper short but complete derivations of all the center series used, in any role, for the tests that were performed.

In this paper we present several comparisons between the speed of convergence of DP Fourier series and that of the corresponding first-order center series, in order to evaluate the relative efficiency with which each series in the pair represents their common limiting function. Corresponding comparisons are made also for the corresponding Fourier Conjugate (FC) series and their limiting functions. All the concepts and results involved, as well as the underlying theory, were developed in the two aforementioned papers [1] and [2]. We direct the reader to the first one for the definition and discussion of concepts and notations, while most of the center series tested in this paper were first derived in the appendices of the second one. The short derivations of all the center series used in this paper can be found in Appendix A.

The center series evaluated here are those obtained from the corresponding DP Fourier series by a single factorization of the dominant singularities of $w(z)$ on the unit circle, which we will refer to as the first-order center series. However, in order to execute the comparative test of the first-order center series and the corresponding DP Fourier series, in some cases it was necessary to use the second-order center series, obtained by a double factorization of the dominant singularities, as tools to produce high-precision numerical representations of the limiting functions. These high-precision representations were then used as gauges for the tests with both types of series.

The efficiency measured and reported is meant in a mathematical sense rather than a technically computational sense. It is measured in terms of the number of terms of the series which must be added up to yield a certain predetermined level of precision in the results. We do, however, report the approximate processing time spent by each run of the programs used for the measurements. The technical optimization of the computer code involved was not at all an issue in the numerical tests. The optimization was left to be done automatically by the compiler, in the most usual and standard way.

2 Tests of Center Series

Three different classes of real functions are represented in the tests. Two of the examples chosen are of continuous functions, namely the triangular wave and the parabolic wave, the latter being also differentiable. In these two cases the Fourier series converge quite rapidly, since the Fourier coefficients a_k, with $k = 1, 2, 3, \ldots, \infty$, behave for large values of k as $1/k^2$ in the first case and as $1/k^3$ in the second, with the consequence that the improvement obtained with the use of the center series is modest. All the remaining examples involve discontinuous functions. In the case of the two forms of sawtooth wave and the two forms of square wave the functions are discontinuous, but limited. These series converge much slower that the previous ones, having coefficients that behave as $1/k$ for large values of k, and in this case the use of the first-order center series produces very significant improvements in the rhythm of convergence, and sometimes very large ones.

The two remaining examples are simple series concocted to be convergent almost everywhere but to have coefficients that behave as $1/\sqrt{k}$ for large values of k, and which due to this converge much slower than those in the previous class of discontinuous functions. The resulting functions are not limited, being in fact logarithmically divergent at a single point. In this case the use of the center series produces truly enormous advantages, to the point where it is not even possible to measure the improvement in some cases, due to the enormous time it would take to complete the standard Fourier runs, which could run up to many years of CPU time. In this case the use of the center series, and in particular of the second-order center series, represents the qualitative difference between being able to deal with these functions with numerical ease and not being able to deal with them at all by numerical means.

The numerical tests were executed on a one-dimensional regular lattice of points defining a set of 1000 intervals between $\theta = -\pi$ and $\theta = \pi$, these two points being identified with each other by periodic boundary conditions. The special points, which correspond to the singularities of $w(z)$ on the unit circle, and where the representation of the real functions by the center series is not defined, were excluded from the set of lattice points. At each remaining point the two types of series were added up until the absolute value of the difference between the approximate value obtained and the known exact result fell below the required threshold. Some care was taken to avoid mistaking for convergence the mere accidental passages through the limiting value during oscillations. In the case of the simple piece-wise functions used as examples the exact result is known because we have piece-wise expressions in closed form for these functions. However, in some cases, and in particular for the Fourier Conjugate functions of all the functions used as examples, no such expressions in closed form are available.

In all the cases where exact expressions in closed form for the limiting functions are not available the following strategy was used. First, the limiting function was obtained numerically to a high degree of precision at all the points of the lattice to be used in the tests. The precision target for these calculations was set at $\varepsilon = 10^{-16}$, while the comparison tests were performed for target precisions ranging from $\varepsilon = 10^{-3}$ to $\varepsilon = 10^{-8}$. Of course the operation of adding up the series to such high level of precision typically takes a long time. In fact, in most cases it is practically impossible to do this using the Fourier series. Therefore, we used the first-order center series for some of these preliminary calculations. This worked well in the case of the better-behaved examples, namely for those involving functions which are at least continuous. In all other cases, namely those involving discontinuous functions, even the first-order center series took too long to achieve the desired high level of precision, so that in these cases we used the second-order center series, which in all cases was sufficient for our ends. The derivation of these second-order series can also be found in Appendix A.

In the tables shown in Appendix B we report the number of series terms which were added in each case, for each series and for each level of precision to be achieved. We report two results in each case, the average number of terms added, considering all the valid lattice points in the interval $[-\pi, \pi]$, and the maximum number of terms added at any single point. This usually corresponds to the points right next to a special point, since these in turn correspond to singularities of $w(z)$ on the complex plane. The ratios reported express how much more efficient the first-order center series is, as compared to the original DP Fourier series.

For each example worked out we give explicitly, in the next few sub-sections, the DP Fourier series and the corresponding first-order center series, for both the original function and the corresponding FC function. When it is the case, the second-order center series is also given. The special points are listed explicitly, as well as the value of the original DP function at those points. In the graphs shown in this paper both the original DP function and the FC function are plotted using the results from calculations with first-order center series added up to precision $\varepsilon = 10^{-6}$, on the same graph, in order to illustrate the general behavior of the functions.

The source code for all the programs used for the numerical calculations in this paper is freely available online on the web [3]. The compilation structure which is included with them is meant to work on Linux systems, and all the data reported was produced on a Debian-Linux distribution version 7.6 running on 2.4 GHz AMD64 hardware with 48 GB of RAM and two CPUs. These were Intel quad-core Xeon CPUs with hyper-threading capabilities, but no parallelization of any kind was included in the programs. Therefore,

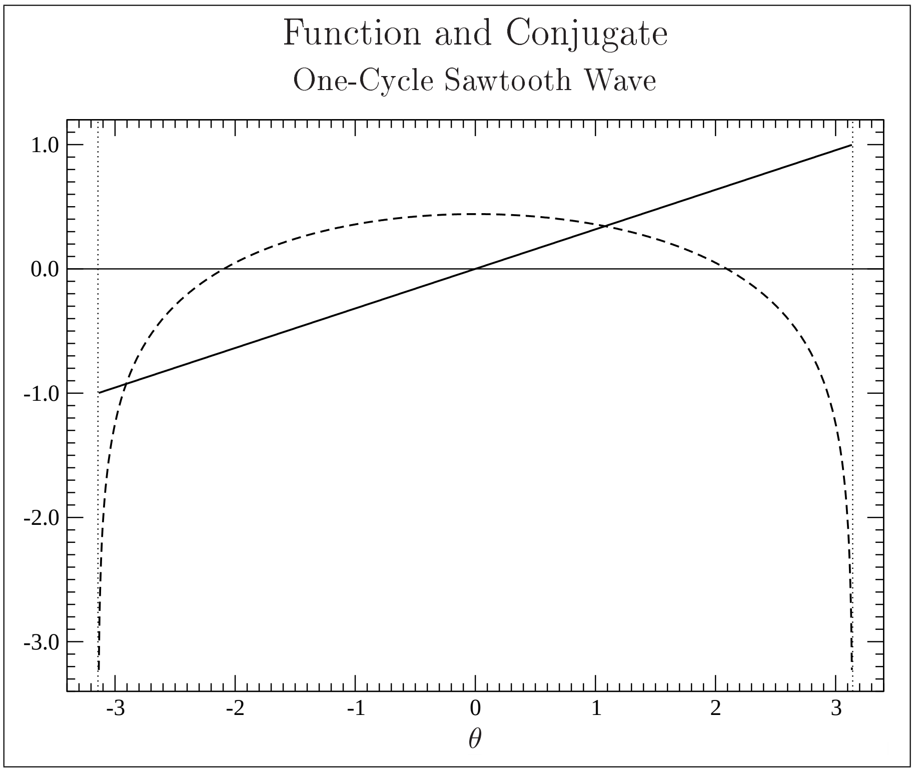

Figure 1: One-cycle sawtooth wave: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special point.

all processing times reported are single-CPU, single-core times. Very little was done by hand in the way of optimization, which was mostly done automatically by the compiler. However, we do report the time of each run, in order to give some idea of the absolute efficiency which may be accomplished in practice with each type of series.

2.1 The One-Cycle Sawtooth Wave

Consider the Fourier series of the one-cycle unit-amplitude sawtooth wave, which is just the linear function θ/π between $-\pi$ and π, and therefore an odd function of θ, as shown in Figure 1. The original function is given by the DP Fourier series

$$f_{\rm s}(\theta) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(k\theta),$$

and the corresponding FC function is given by the DP Fourier series

$$f_{\rm c}(\theta) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cos(k\theta).$$

These two series are convergent almost everywhere, but not absolutely or uniformly convergent. There is a single special point at $\theta = \pm\pi$, where we have for the original function $f_{\rm s}(\pm\pi) = 0$. At this point the original function is discontinuous and the corresponding FC function diverges logarithmically. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{\sin(\theta/2) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\sin[(k+1/2)\theta]\right\},$$

for $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the first-order center series is given by

$$f_{\rm c}(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{\cos(\theta/2) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\cos[(k+1/2)\theta]\right\},$$

for $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The representation of the original function in terms of the second-order center series is given by

$$f_{\rm s}(\theta) = \frac{1}{4\pi\cos^2(\theta/2)}\left\{6\sin(\theta/2)\cos(\theta/2) + \sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,\sin[(k+1)\theta]\right\},$$

for $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the second-order center series is given by

$$f_{\rm c}(\theta) = \frac{1}{4\pi\cos^2(\theta/2)}\left\{-1 + 6\cos^2(\theta/2) + \sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,\cos[(k+1)\theta]\right\},$$

for $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.1 of Appendix A, and the results of the tests are shown in the tables in Subsections B.1 and B.2 of Appendix B.

2.2 The Standard Square Wave

Consider the Fourier series of the standard unit-amplitude square wave, which is an odd function of θ, as shown in Figure 2. The original function is given by the DP Fourier series

$$f_{\rm s}(\theta) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{k}\,\sin(k\theta),$$

where $k = 2j+1$, and the corresponding FC function is given by the DP Fourier series

$$f_{\rm c}(\theta) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{k}\,\cos(k\theta),$$

where $k = 2j+1$. These two series are convergent almost everywhere, but not absolutely or uniformly convergent. There are two special points at $\theta = 0$ and at $\theta = \pm\pi$, where we have for the original function $f_{\rm s}(0) = 0$ and $f_{\rm s}(\pm\pi) = 0$. At these points the original function is discontinuous and the corresponding FC function diverges logarithmically. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{2}{\pi\sin(\theta)}\left\{1 - \sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\cos[(k+1)\theta]\right\},$$

where $k = 2j+1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the first-order center series is given by

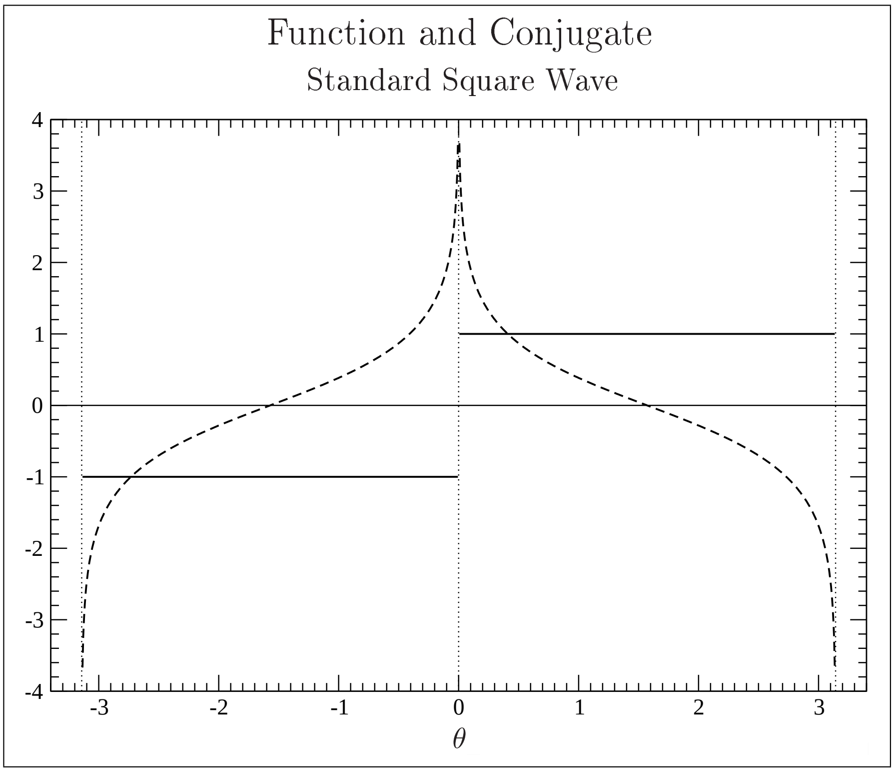

Figure 2: Standard square wave: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special points.

$$f_{\rm c}(\theta) = \frac{2}{\pi\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\sin[(k+1)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The representation of the original function in terms of the second-order center series is given by

$$f_{\rm s}(\theta) = \frac{2}{3\pi\sin^2(\theta)}\left\{4\sin(\theta) - \sum_{j=0}^{\infty}\frac{12}{k(k+2)(k+4)}\,\sin[(k+2)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the second-order center series is given by

$$f_{\rm c}(\theta) = \frac{2}{3\pi\sin^2(\theta)}\left\{\cos(\theta) - \sum_{j=0}^{\infty}\frac{12}{k(k+2)(k+4)}\,\cos[(k+2)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.2 of Appendix A, and the results of the tests are shown in the tables in Subsections B.3 and B.4 of Appendix B.

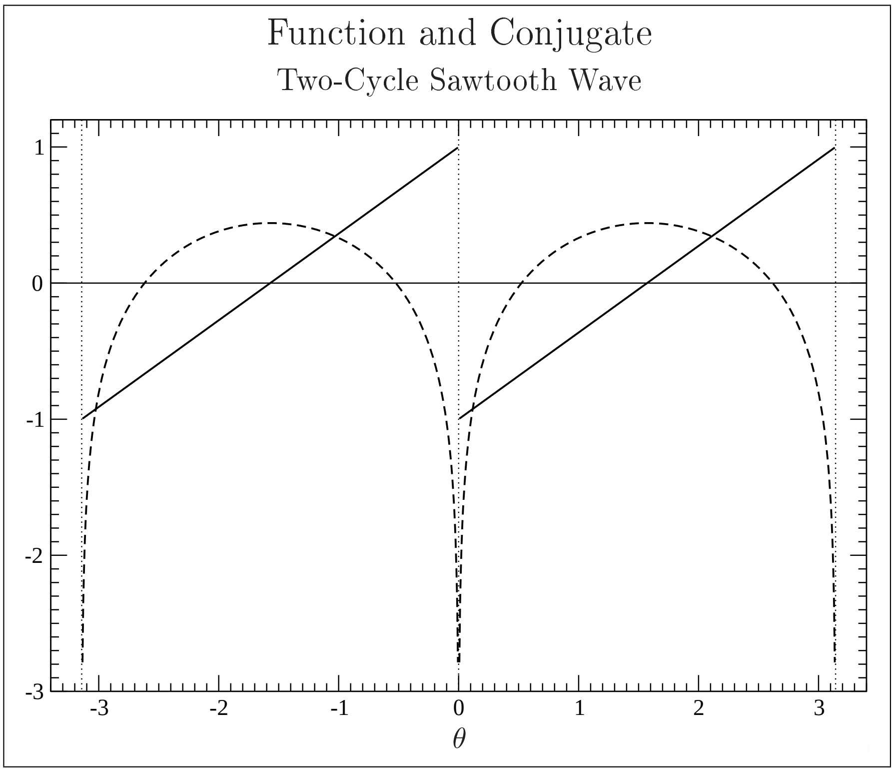

Figure 3: Two-cycle sawtooth wave: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special points.

2.3 The Two-Cycle Sawtooth Wave

Consider the Fourier series of the two-cycle unit-amplitude sawtooth wave, which is an odd function of θ, as shown in Figure 3. The original function is given by the DP Fourier series

$$f_{\rm s}(\theta) = -\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{k} \sin(k\theta),$$

where $k = 2j$, and the corresponding FC function is given by the DP Fourier series

$$f_{\rm c}(\theta) = -\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{1}{k} \cos(k\theta),$$

where $k = 2j$. These two series are convergent almost everywhere, but not absolutely or uniformly convergent. There are two special points at $\theta = 0$ and at $\theta = \pm\pi$, where we have for the original function $f_{\rm s}(0) = 0$ and $f_{\rm s}(\pm\pi) = 0$. At these points the original function is discontinuous and the corresponding FC function diverges logarithmically. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{1}{\pi \sin(\theta)} \left\{ -\cos(\theta) + \sum_{j=1}^{\infty} \frac{4}{k(k+2)} \cos[(k+1)\theta] \right\},$$

where $k = 2j$, for $\theta \neq 0$ and $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the first-order center series is given by

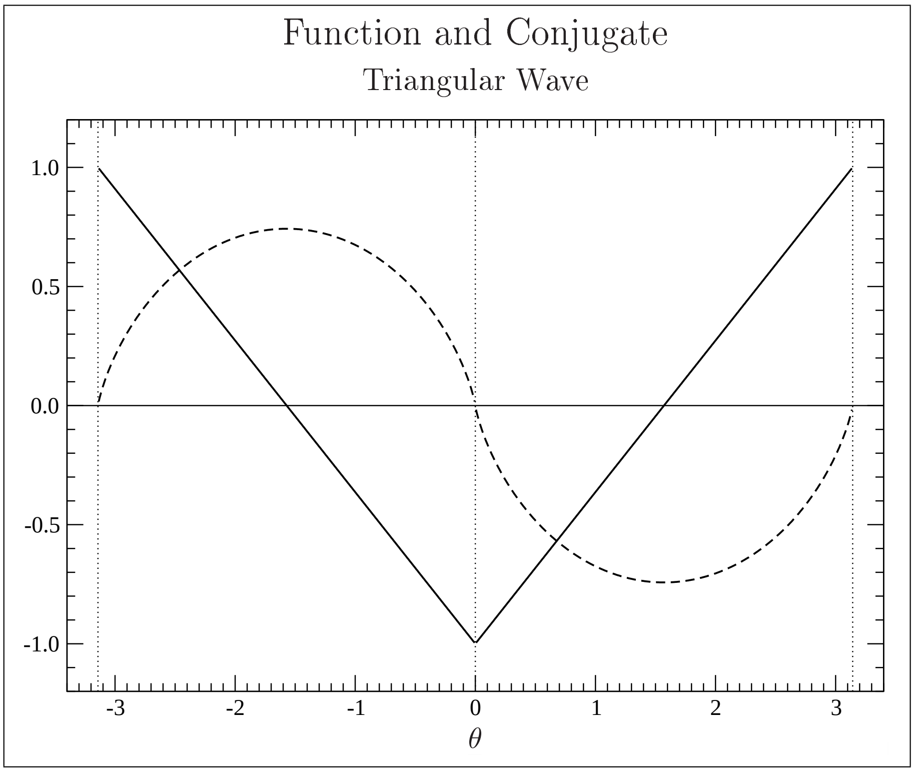

Figure 4: Triangular wave: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special points.

$$f_{\rm c}(\theta) = \frac{1}{\pi\sin(\theta)} \left\{ \sin(\theta) - \sum_{j=1}^{\infty} \frac{4}{k(k+2)} \sin[(k+1)\theta] \right\},$$

where $k = 2j$, for $\theta \neq 0$ and $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The representation of the original function in terms of the second-order center series is given by

$$f_{\rm s}(\theta) = \frac{1}{4\pi\sin^2(\theta)} \left\{ -6\sin(\theta)\cos(\theta) + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)} \sin[(k+2)\theta] \right\},$$

where $k = 2j$, for $\theta \neq 0$ and $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the second-order center series is given by

$$f_{\rm c}(\theta) = \frac{1}{4\pi\sin^2(\theta)} \left\{ -1 + 6\sin^2(\theta) + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)} \cos[(k+2)\theta] \right\},$$

where $k = 2j$, for $\theta \neq 0$ and $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.3 of Appendix A, and the results of the tests are shown in the tables in Subsections B.5 and B.6 of Appendix B.

2.4 The Triangular Wave

Consider the Fourier series of the unit-amplitude triangular wave, which is an even function of θ, as shown in Figure 4. The original function is given by the DP Fourier series

$$f_{\rm c}(\theta) = -\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{k^2}\cos(k\theta),$$

where $k = 2j + 1$, and the corresponding FC function is given by the DP Fourier series

$$f_{\rm s}(\theta) = -\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{k^2}\sin(k\theta),$$

where $k = 2j+1$. These two series are absolutely and uniformly convergent. There are two special points at $\theta = 0$ and at $\theta = \pm\pi$, where we have for the original function $f_{\rm c}(0) = -1$ and $f_{\rm c}(\pm\pi) = 1$. At these points both the original function and the corresponding FC function function are non-differentiable. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm c}(\theta) = -\frac{4}{\pi^2\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\sin[(k+1)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{4}{\pi^2\sin(\theta)}\left\{-1+\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\cos[(k+1)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.4 of Appendix A, and the results of the tests are shown in the tables in Subsections B.7 and B.8 of Appendix B.

2.5 The Shifted Square Wave

Consider the Fourier series of the unit-amplitude square wave, shifted along the θ axis to θ', with $\theta - \theta' = \pi/2$, so that it becomes an even function of θ, as shown in Figure 5. The original function is given by the DP Fourier series

$$f_{\rm c}(\theta) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\cos(k\theta),$$

where $k = 2j + 1$, and the corresponding FC function is given by the DP Fourier series

$$f_{\rm s}(\theta) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\sin(k\theta),$$

where $k = 2j+1$. These two series are convergent almost everywhere, but not absolutely or uniformly convergent. There are two special points at $\theta = \pi/2$ and at $\theta = -\pi/2$, where we have for the original function $f_{\rm c}(\pi/2) = 0$ and $f_{\rm c}(-\pi/2) = 0$. At these points the original

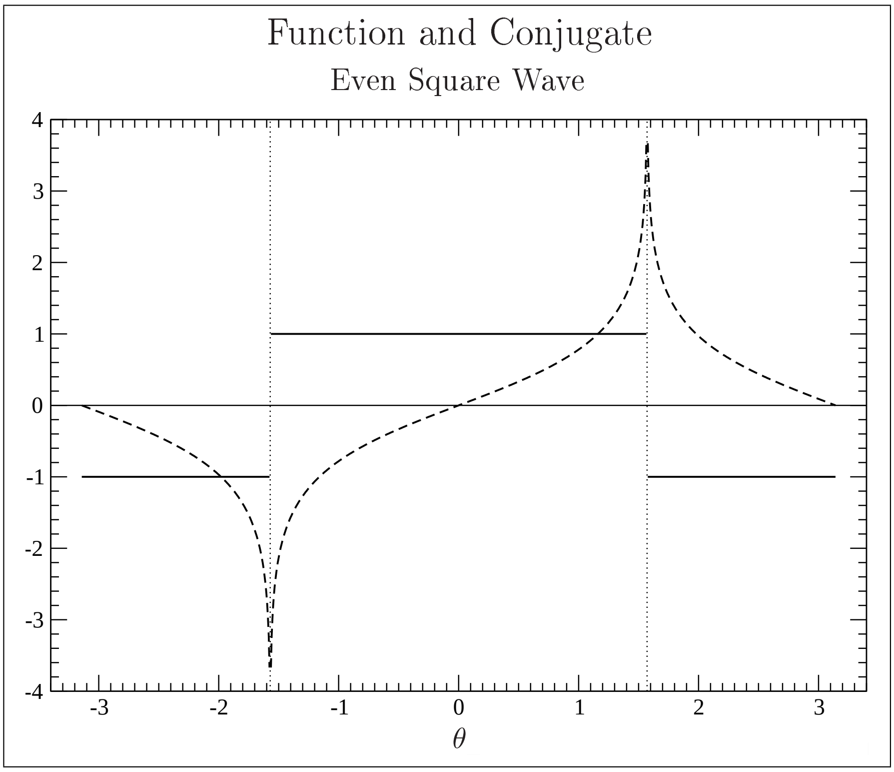

Figure 5: Shifted square wave: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special points.

function is discontinuous and the corresponding FC function diverges logarithmically. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm c}(\theta) = \frac{2}{\pi\cos(\theta)}\left\{1 + \sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\cos[(k+1)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq \pi/2$ and $\theta \neq -\pi/2$, and the representation of the corresponding FC function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{2}{\pi\cos(\theta)}\left\{\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\sin[(k+1)\theta]\right\},$$

where $k = 2j+1$, for $\theta \neq \pi/2$ and $\theta \neq -\pi/2$. These two series are absolutely and uniformly convergent. The representation of the original function in terms of the second-order center series is given by

$$f_{\rm c}(\theta) = \frac{2}{3\pi\cos^2(\theta)}\left\{4\cos(\theta) + \sum_{j=0}^{\infty}\frac{12(-1)^j}{k(k+2)(k+4)}\cos[(k+2)\theta]\right\},$$

where $k = 2j + 1$, for $\theta \neq \pi/2$ and $\theta \neq -\pi/2$, and the representation of the corresponding FC function in terms of the second-order center series is given by

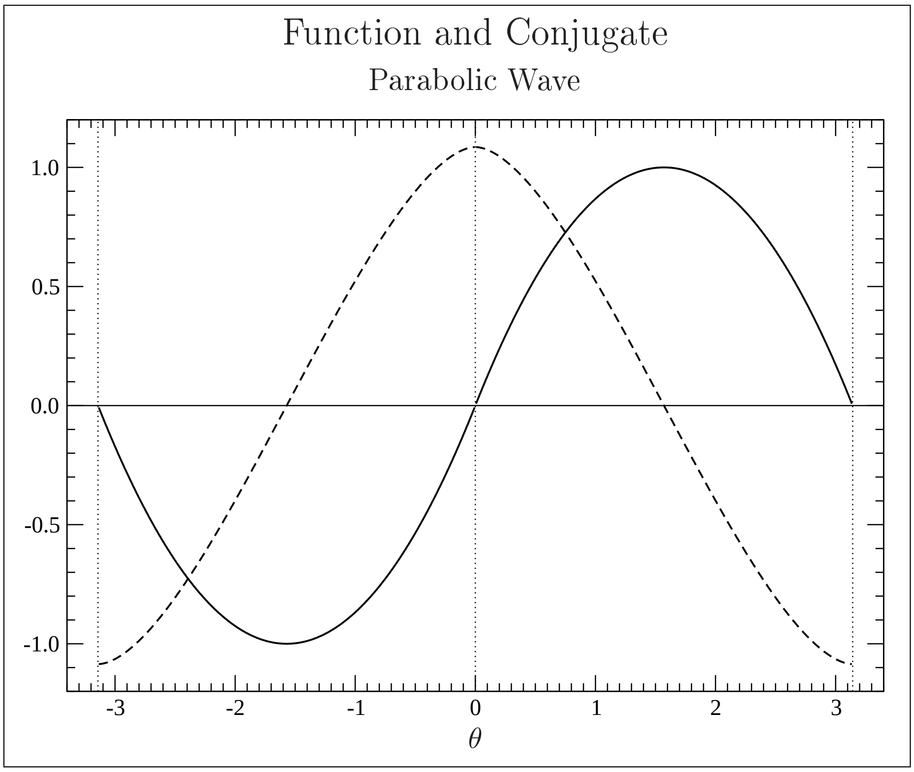

Figure 6: Parabolic wave: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special points.

$$f_{\rm s}(\theta) = \frac{2}{3\pi\cos^2(\theta)} \left\{ \sin(\theta) + \sum_{j=0}^{\infty} \frac{12(-1)^j}{k(k+2)(k+4)} \sin[(k+2)\theta] \right\},$$

where $k = 2j + 1$, for $\theta \neq \pi/2$ and $\theta \neq -\pi/2$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.5 of Appendix A, and the results of the tests are shown in the tables in Subsections B.9 and B.10 of Appendix B.

2.6 The Parabolic Wave

Consider the Fourier series of a unit-amplitude periodic function built with segments of quadratic functions, joined together so that the resulting function is continuous and differentiable, in such a way that the result is an odd function of θ, as shown in Figure 6. The original function is given by the DP Fourier series

$$f_{\rm s}(\theta) = \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{1}{k^3} \sin(k\theta),$$

where $k = 2j + 1$, and the corresponding FC function is given by the DP Fourier series

$$f_{\rm c}(\theta) = \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{1}{k^3} \cos(k\theta),$$

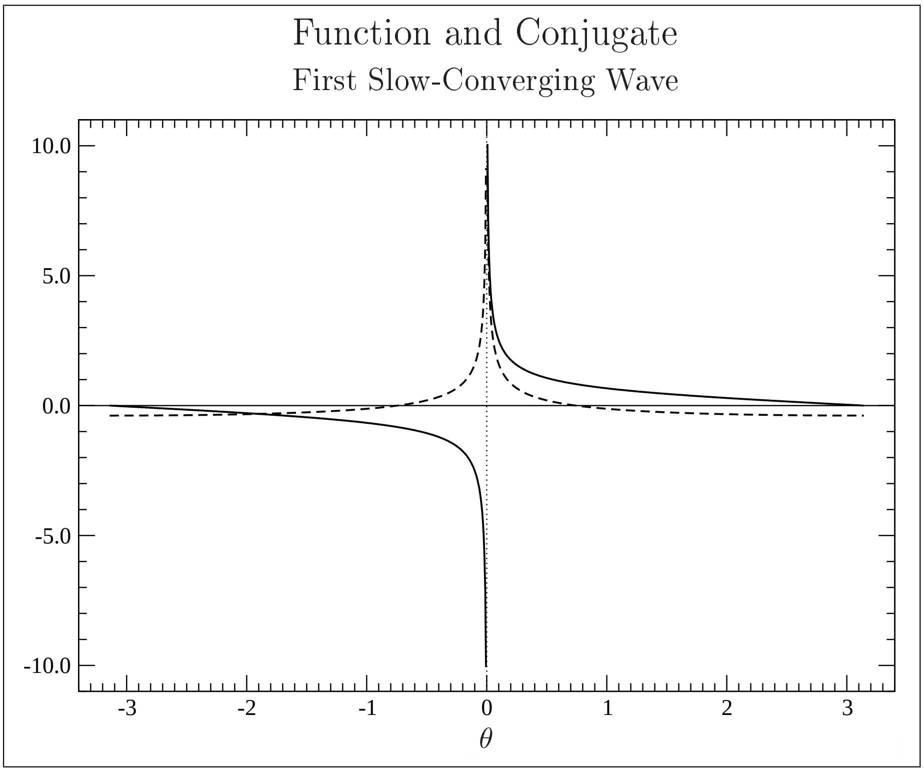

Figure 7: First slow-converging case: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi,\pi]$. The dotted line marks the special point.

where $k = 2j+1$. These two series are absolutely and uniformly convergent. There are two special points at $\theta = 0$ and at $\theta = \pm\pi$, where we have for the original function $f_{\rm s}(0) = 0$ and $f_{\rm s}(\pm\pi) = 0$. At these points both the original function and the corresponding FC function have singularities on their second derivatives. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{16}{\pi^3 \sin(\theta)} \left\{ 1 - \sum_{j=0}^{\infty} \frac{6k(k+2)+8}{k^3(k+2)^3} \cos[(k+1)\theta] \right\},$$

where $k = 2j+1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the first-order center series is given by

$$f_{\rm c}(\theta) = \frac{16}{\pi^3 \sin(\theta)} \left\{ \sum_{j=0}^{\infty} \frac{6k(k+2)+8}{k^3(k+2)^3} \sin[(k+1)\theta] \right\},$$

where $k = 2j+1$, for $\theta \neq 0$ and $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.6 of Appendix A, and the results of the tests are shown in the tables in Subsections B.11 and B.12 of Appendix B.

2.7 The First Slow-Converging Case

The function we will adopt as our original function, which is an odd function of θ, is shown in Figure 7. It is given by the DP Fourier series

$$f_{\rm s}(\theta) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\,\sin(k\theta),$$

and the corresponding FC function is then given by the DP Fourier series

$$f_{\rm c}(\theta) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\,\cos(k\theta).$$

These two series are convergent almost everywhere, but not absolutely or uniformly convergent. There is a single special point at $\theta = 0$, where we have for the original function $f_{\rm s}(0) = 0$. However, the function is not continuous at this point, and its lateral limits to it diverge to $\pm\infty$. In fact, at this point both the original function and the corresponding FC function diverge logarithmically. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{1}{\pi\sin(\theta/2)}\left\{\cos(\theta/2) - \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\cos[(k+1/2)\theta]\right\},$$

for $\theta \neq 0$, and the representation of the corresponding FC function in terms of the first-order center series is given by

$$f_{\rm c}(\theta) = \frac{1}{\pi\sin(\theta/2)}\left\{-\sin(\theta/2) + \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\sin[(k+1/2)\theta]\right\},$$

for $\theta \neq 0$. These two series are absolutely and uniformly convergent. The representation of the original function in terms of the second-order center series is given by

$$\begin{aligned} f_{\rm s}(\theta) \;\;=\;\; & \frac{1}{4\pi\sin^2(\theta/2)}\times \\ & \times\left\{\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)+\right. \\ & \left. -\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,\sin[(k+1)\theta]\right\}, \end{aligned}$$

for $\theta \neq 0$, and the representation of the corresponding FC function in terms of the second-order center series is given by

$$\begin{aligned} f_{\rm c}(\theta) \;\;=\;\; & \frac{1}{4\pi\sin^2(\theta/2)}\times \\ & \times\left\{\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\sin^2(\theta/2)+\right. \\ & \left. -\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,\cos[(k+1)\theta]\right\}, \end{aligned}$$

for $\theta \neq 0$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.7 of Appendix A, and the results of the tests are shown in the tables in Subsections B.13 and B.14 of Appendix B.

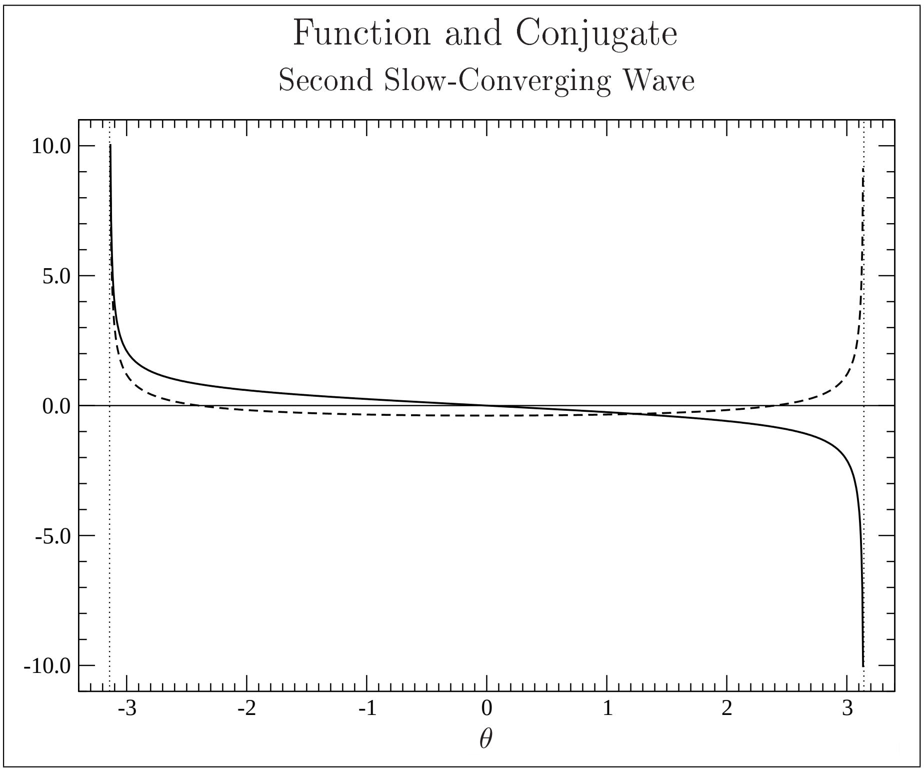

Figure 8: Second slow-converging case: the original function (solid line) and its conjugate function (dashed line) plotted within the periodic interval $[-\pi, \pi]$. The dotted lines mark the special point.

2.8 The Second Slow-Converging Case

The function we will adopt as our original function, which is an odd function of θ, is shown in Figure 8. It is given by the DP Fourier series

$$f_{\rm s}(\theta) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \sin(k\theta),$$

and the corresponding FC function is then given by the DP Fourier series

$$f_{\rm c}(\theta) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \cos(k\theta).$$

These two series are convergent almost everywhere, but not absolutely or uniformly convergent. There is a single special point at $\theta = \pm\pi$, where we have for the original function $f_{\rm s}(\pm\pi) = 0$. However, the function is not continuous at this point, and its lateral limits to it diverge to $\pm\infty$. In fact, at this point both the original function and the corresponding FC function diverge logarithmically. The representation of the original function in terms of the first-order center series is given by

$$f_{\rm s}(\theta) = \frac{1}{\pi \cos(\theta/2)} \left\{ -\sin(\theta/2) + \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}(k+1) + k\sqrt{k+1}} \sin[(k+1/2)\theta] \right\},$$

for $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the first-order center series is given by

$$f_{\mathrm{c}}(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{-\cos(\theta/2) + \sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\cos[(k+1/2)\theta]\right\},$$

for $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The representation of the original function in terms of the second-order center series is given by

$$\begin{aligned} f_{\mathrm{s}}(\theta) &= \frac{1}{4\pi\cos^2(\theta/2)}\times \\ &\times\left\{-\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)+\right. \\ &\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,(-1)^k\sin[(k+1)\theta]\right\}, \end{aligned}$$

for $\theta \neq \pm\pi$, and the representation of the corresponding FC function in terms of the second-order center series is given by

$$\begin{aligned} f_{\mathrm{c}}(\theta) &= \frac{1}{4\pi\cos^2(\theta/2)}\times \\ &\times\left\{\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\cos^2(\theta/2)+\right. \\ &\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,(-1)^k\cos[(k+1)\theta]\right\}, \end{aligned}$$

for $\theta \neq \pm\pi$. These two series are absolutely and uniformly convergent. The derivation of the center series can be found in Section A.8 of Appendix A, and the results of the tests are shown in the tables in Subsections B.15 and B.16 of Appendix B.

3 Summary of the Numerical Results

The numerical results obtained show that the improvement in convergence speed obtained with the center series depends first and foremost on whether or not the Fourier series involved are absolutely and uniformly convergent. This is as one would expect on general grounds. For example, in the case of the parabolic wave, which is continuous and differentiable, with Fourier coefficients a_k that behave as $1/k^3$ for large values of k, so that the Fourier series is absolutely and uniformly convergent, the improvement of the ratio of the average number of added terms varies from about 1.2 to about 2.8, depending on the precision lever required. In the case of the triangular wave, which is continuous but not differentiable, with Fourier coefficients that behave as $1/k^2$ for large values of k, so that the Fourier series is still absolutely and uniformly convergent, the improvement of the same ratio is larger, varying from about 1.9 to about 11.6.

In the cases of the discontinuous functions with Fourier coefficients that behave as $1/k$ for large values of k, so that the Fourier series is point-wise convergent almost everywhere but not absolutely or uniformly convergent, the ratio varies from about 21 to about 6600, being therefore very significant, specially for the higher levels of precision. In the case of the functions associated to the slow-converging series, with Fourier coefficients that behave as $1/\sqrt{k}$ for large values of k, so that the Fourier series is also point-wise convergent almost everywhere but not absolutely or uniformly convergent, the same ratio varies from about 1.0×10^4 to about 4.8×10^6, where we consider only the three lower levels of precision, which were the only ones for which we were able to run the Fourier programs within a feasible amount of time.

Function	a_0	a_1	a_2
One-cycle sawtooth wave	0.6287	1.1530	0.7668
Its conjugate function	0.6259	1.1533	0.9987
Standard square wave	0.6378	1.1553	0.6206
Its conjugate function	0.6388	1.1547	0.7636
Two-cycle sawtooth wave	0.6446	1.1526	0.8442
Its conjugate function	0.6399	1.1530	1.0229
Triangular wave	0.5126	0.3863	0.2718
Its conjugate function	0.4710	0.3958	0.3994
Shifted square wave	0.6380	1.1552	0.6105
Its conjugate function	0.6389	1.1547	0.7585
Parabolic wave	0.4836	0.2016	0.3594
Its conjugate function	0.5442	0.1941	0.2319
First slow-converging wave	N/A	N/A	N/A
Its conjugate function	0.9695	3.0811	223.53
Second slow-converging wave	1.0041	3.0761	241.09
Its conjugate function	0.9693	3.0812	223.75

Table 1: Coefficients of the exponential fits to the ratios of the average numbers of added terms, as functions o the target precision.

In short, if the original DP Fourier series is already absolutely and uniformly convergent, as in the first two cases above, then the improvement obtained is modest. Otherwise, since the center series is always absolutely and uniformly convergent, the improvement obtained is large. On a finer scale, there is always some improvement when the first-order center series is used, and further improvement with the second-order one. This is a consequence of the fact that every factorization of the dominant singularities adds a factor of k to the denominator of the coefficients of the center series. We see therefore that the greater improvements come when the coefficients go from the $1/k$ or $1/k^{(1/2)}$ behavior in the Fourier case to the $1/k^2$ or $1/k^{(3/2)}$ behavior in the center series case, so that the original series is not absolutely or uniformly convergent, while the corresponding center series is.

In all cases except the very slow-converging ones the ratios of the average number of added terms can be fit very well by increasing exponentials, as functions of the logarithms of the target precision ε,

$$r = a_0 \, \mathrm{e}^{a_1 \xi} + a_2, \tag{1}$$

where r is the ratio for the average number of added terms and $\xi = -\log_{10}(\varepsilon)$. The coefficients a_0, a_1 and a_2 are always positive and of the order of one, and the correlation coefficients of the fits are very close to one, while the mean square error is of the order of about 1% or less. The coefficients obtained for the parameters of these fits can be seen in Table 1. As one can see there, the fits were also worked out for some of the very slow-converging cases. In these cases, since we had only three data points available and three constants to fit, the fits were, of course, exact ones. The only possible justification for our producing these fits is, of course, that they work so well on the other cases. Using these fits one can estimate that, under the circumstances in which we executed the runs, the Fortran runs for the slow-converging cases, for a precision of 10^{-8}, would take something like a few tens of thousands of years to complete. This is to be compared to the 30 seconds or so that it took to run the center series for these functions, with the same precision.

We see therefore that with the utilization of the coefficients in Table 1 the formula in Equation 1 above can be used as a comparative performance predictor for higher levels of precision. This should work well for the average number of added terms, and may also give some idea of the running time, if one takes into consideration the hardware involved. However, it will probably produce an underestimation of the running time in the case of the higher precision cases, since it is observed that in these cases the standard Fourier runs tend to take a disproportionally large amount of time to complete. This is probably related to technical optimization issues and may depend strongly on the compiler used and on the details of the hardware.

4 Conclusions

We may conclude that it is always computationally advantageous to use the center series instead of the corresponding DP Fourier series. In the case of the slower-converging DP Fourier series the improvement in the speed of convergence can be very large indeed, to the point where it may represent a qualitative difference in our ability to deal with the function in a practical manner. With the use of a sufficiently high-order center series, essentially any real function that gives origin to a set of finite DP Fourier coefficients, and that results in an inner analytic function that has a finite number of sufficiently soft singularities on the unit circle, can be very well represented numerically by that center series.

The one limitation to the use of the center series is the need to know, in order to derive the form of the coefficients of the series, the positions of all the dominant singularities of the corresponding inner analytic function on the unit circle. However, the construction process is safe, in the sense that in the worst case scenario all that happens is that one fails to obtain a better convergence speed. Moreover, this fact can be verified analytically during the construction of the center series, before any computer time is actually spent. In effect, the fact of the failure to obtain improvement may itself serve as a guide to search for the correct positions of the singularities, by what are essentially algebraic means, as explained in [2].

The position of the singularities can be induced by the qualitative analytical properties of the function to be represented, such as that it is discontinuous or non-differentiable at certain points. In physics applications these characteristics are bound to be reflected in the structure of the problem being dealt with, so that a physical analysis may suffice to determine the singularities. In any case, whenever it is possible to use them, the center series constitute a significant improvement in our ability to represent real functions numerically in practical applications.

5 Acknowledgements

The author would like to thank the numerical group of the Department of Mathematical Physics of the Physics Institute of the University of São Paulo for the use of its high-performance computational infrastructure.

A Derivations of Center Series

In this appendix we give short but complete derivations of all the center series used in this paper. In each case we start with the complex power series S_z related to the original DP Fourier series, and construct from it the complex center series C_z. This involves the positions of the singularities of the corresponding inner analytic function $w(z)$ on the unit circle. We then write S_z in terms of C_z and take the real and imaginary parts, in order to obtain the center series corresponding to the original DP Fourier series and of its FC series.

One must keep in mind that the final forms obtained for the functions $f_{\rm c}(\theta)$ and $f_{\rm s}(\theta)$ in terms of the center series are only valid away from the special points on the periodic interval, which correspond to the singular points of $w(z)$ on the complex plane. The values of the DP real functions at these special points are usually determined very easily by the original Fourier series.

Starting from eight Fourier Conjugate pairs of DP Fourier series, we derive eight pairs of first-order center series and six pairs of second-order center series. Of these 44 series a total of 40 series were used in this paper, of which 16 DP Fourier series and 16 first-order center series were tested against each other.

A.1 The One-Cycle Sawtooth Wave

Consider the one-cycle unit-amplitude sawtooth wave, given by the sine series

$$S_{\rm s} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(k\theta).$$

The corresponding FC series is then

$$S_{\rm c} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cos(k\theta),$$

and the complex power series S_z is given by

$$S_z = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k,$$

of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.1.1 First-Order Center Series

There is a single dominant singularity at $z = -1$, so that we must use a single factor of $(z+1)$ in the construction of the first-order center series,

$$S_z = \frac{1}{z+1} C_z,$$

where

$$\begin{aligned} C_z &= \frac{2}{\pi} (z+1) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k \\ &= \frac{2}{\pi} z \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)} z^k \right], \end{aligned}$$

where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{2}{\pi}\,\frac{z}{z+1}\left[1+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,z^k\right].$$

In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{z+1} = \frac{1}{2} + \frac{\imath}{2}\,\frac{\sin(\theta)}{1+\cos(\theta)}.$$

If we write this in terms of $\theta/2$ we get

$$\frac{z}{z+1} = \frac{1}{2} + \frac{\imath}{2}\,\frac{\sin(\theta/2)}{\cos(\theta/2)},$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{1}{\pi}\left[1+\imath\frac{\sin(\theta/2)}{\cos(\theta/2)}\right]\left[1+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\cos(k\theta)+\imath\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\sin(k\theta)\right] \\ &= \frac{1}{\pi\cos(\theta/2)}\left\{\cos(\theta/2)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\cos[(k+1/2)\theta]+\right. \\ &\qquad \left. +\,\imath\sin(\theta/2)+\imath\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\sin[(k+1/2)\theta]\right\}, \end{aligned}$$

where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{\sin(\theta/2)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\sin[(k+1/2)\theta]\right\},$$

and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{\cos(\theta/2)+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,\cos[(k+1/2)\theta]\right\}.$$

A.1.2 Second-Order Center Series

There is a single dominant singularity at $z = -1$, so that we must use a factor of $(z+1)^2$ in the construction of the second-order center series,

$$S_z = \frac{1}{(z+1)^2}\,C_z,$$

where

$$
\begin{aligned}
C_z &= \frac{2}{\pi}\,(z+1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\,z^k \\
&= \frac{1}{\pi}\,z\left[2+3z+\sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,z^{k+1}\right],
\end{aligned}
$$

where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^3$ when $k\to\infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$
S_z = \frac{1}{\pi}\,\frac{z}{(z+1)^2}\left[2+3z+\sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,z^{k+1}\right].
$$

In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z=\rho\exp(\imath\theta)$ we have on the unit circle

$$
\frac{z}{(z+1)^2} = \frac{1}{2[1+\cos(\theta)]}.
$$

If we write this in terms of $\theta/2$ we get

$$
\frac{z}{(z+1)^2} = \frac{1}{4\cos^2(\theta/2)}.
$$

We also have that

$$
\begin{aligned}
2+3z &= [2+3\cos(\theta)]+\imath[3\sin(\theta)] \\
&= [-1+6\cos^2(\theta/2)]+\imath[6\sin(\theta/2)\cos(\theta/2)],
\end{aligned}
$$

and therefore we have for S_z on the unit circle

$$
\begin{aligned}
S_z &= \frac{1}{4\pi\cos^2(\theta/2)}\left\{-1+6\cos^2(\theta/2)+\sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,\cos[(k+1)\theta]+\right. \\
&\qquad \left. +\imath 6\sin(\theta/2)\cos(\theta/2)+\imath\sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,\sin[(k+1)\theta]\right\}.
\end{aligned}
$$

where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$
f_{\rm s}(\theta) = \frac{1}{4\pi\cos^2(\theta/2)}\left\{6\sin(\theta/2)\cos(\theta/2)+\sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,\sin[(k+1)\theta]\right\},
$$

and the corresponding FC function is given by the real part,

$$
f_{\rm c}(\theta) = \frac{1}{4\pi\cos^2(\theta/2)}\left\{-1+6\cos^2(\theta/2)+\sum_{k=1}^{\infty}\frac{4(-1)^{k+1}}{k(k+1)(k+2)}\,\cos[(k+1)\theta]\right\}.
$$

A.2 The Standard Square Wave

Consider the standard unit-amplitude square wave, given by the sine series

$$S_{\rm s} = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{k} \,\sin(k\theta),$$

where $k = 2j + 1$. The corresponding FC series is then

$$S_{\rm c} = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{k} \,\cos(k\theta),$$

where $k = 2j + 1$, and the complex power series S_z is given by

$$S_z = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{k} \,z^k,$$

where $k = 2j + 1$, of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.2.1 First-Order Center Series

There are two dominant singularities, located at $z = 1$ and at $z = -1$, so that we must use factors of $(z - 1)(z + 1) = z^2 - 1$ in the construction of the first-order center series,

$$S_z = \frac{1}{z^2 - 1} \,C_z,$$

where

$$\begin{aligned} C_z &= \frac{4}{\pi} \left(z^2 - 1\right) \sum_{j=0}^{\infty} \frac{1}{k} \,z^k \\ &= \frac{4}{\pi} \,z \left[-1 + \sum_{j=0}^{\infty} \frac{2}{k(k + 2)} \,z^{k+1} \right], \end{aligned}$$

where $k = 2j + 1$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{4}{\pi} \frac{z}{z^2 - 1} \left[-1 + \sum_{j=0}^{\infty} \frac{2}{k(k + 2)} \,z^{k+1} \right],$$

where $k = 2j + 1$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho \exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{z^2 - 1} = -\frac{\imath}{2 \sin(\theta)},$$

and therefore we have for S_z on the unit circle

$$
\begin{aligned}
S_z &= -\frac{2}{\pi}\,\frac{\imath}{\sin(\theta)}\left\{-1+\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\cos[(k+1)\theta]+\imath\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\sin[(k+1)\theta]\right\}\\
&= \frac{2}{\pi\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\sin[(k+1)\theta]+\imath-\imath\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\cos[(k+1)\theta]\right\},
\end{aligned}
$$

where $k=2j+1$, and where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$
f_{\rm s}(\theta)=\frac{2}{\pi\sin(\theta)}\left\{1-\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\cos[(k+1)\theta]\right\},
$$

where $k=2j+1$, and the corresponding FC function is given by the real part,

$$
f_{\rm c}(\theta)=\frac{2}{\pi\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{2}{k(k+2)}\,\sin[(k+1)\theta]\right\},
$$

where $k=2j+1$.

A.2.2 Second-Order Center Series

There are two dominant singularities, located at $z=1$ and at $z=-1$, so that we must use factors of $(z-1)^2(z+1)^2=\left(z^2-1\right)^2$ in the construction of the second-order center series,

$$
S_z=\frac{1}{\left(z^2-1\right)^2}\,C_z,
$$

where

$$
\begin{aligned}
C_z &= \frac{4}{\pi}\left(z^2-1\right)^2\sum_{j=0}^{\infty}\frac{1}{k}\,z^k\\
&= \frac{4}{3\pi}\,z\left[3-5z^2+\sum_{j=0}^{\infty}\frac{24}{k(k+2)(k+4)}\,z^{k+3}\right],
\end{aligned}
$$

where $k=2j+1$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^3$ when $k\to\infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$
S_z=\frac{4}{3\pi}\,\frac{z}{\left(z^2-1\right)^2}\left[3-5z^2+\sum_{j=0}^{\infty}\frac{24}{k(k+2)(k+4)}\,z^{k+3}\right],
$$

where $k=2j+1$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z=\rho\exp(\imath\theta)$ we have on the unit circle

$$
\frac{z}{\left(z^2-1\right)^2}=-\,\frac{z^*}{4\sin^2(\theta)},
$$

where $z^* z = 1$. We also have that

$$\begin{aligned} z^*\left(3 - 5z^2\right) &= 3z^* - 5z \\ &= [-2\cos(\theta)] + \imath[-8\sin(\theta)], \end{aligned}$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{2}{3\pi\sin^2(\theta)}\left\{\cos(\theta) - \sum_{j=0}^{\infty}\frac{12}{k(k+2)(k+4)}\cos[(k+2)\theta] + \right. \\ &\qquad \left. + \imath 4\sin(\theta) - \imath\sum_{j=0}^{\infty}\frac{12}{k(k+2)(k+4)}\sin[(k+2)\theta]\right\}, \end{aligned}$$

where $k = 2j + 1$, and where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{2}{3\pi\sin^2(\theta)}\left\{4\sin(\theta) - \sum_{j=0}^{\infty}\frac{12}{k(k+2)(k+4)}\sin[(k+2)\theta]\right\},$$

where $k = 2j + 1$, and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta) = \frac{2}{3\pi\sin^2(\theta)}\left\{\cos(\theta) - \sum_{j=0}^{\infty}\frac{12}{k(k+2)(k+4)}\cos[(k+2)\theta]\right\},$$

where $k = 2j + 1$.

A.3 The Two-Cycle Sawtooth Wave

Consider the two-cycle unit-amplitude sawtooth wave, given by the sine series

$$S_{\rm s} = -\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{k}\sin(k\theta),$$

where $k = 2j$. The corresponding FC series is then

$$S_{\rm c} = -\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{k}\cos(k\theta),$$

where $k = 2j$, and the complex power series S_z is given by

$$S_z = -\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{k}z^k,$$

where $k = 2j$, of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.3.1 First-Order Center Series

There are two dominant singularities, located at $z = 1$ and at $z = -1$, so that we must use factors of $(z-1)(z+1) = z^2 - 1$ in the construction of the first-order center series,

$$S_z = \frac{1}{z^2 - 1} C_z,$$

where

$$\begin{aligned} C_z &= -\frac{4}{\pi}\,(z^2 - 1)\sum_{j=1}^{\infty}\frac{1}{k}\,z^k \\ &= \frac{2}{\pi}\,z^2\left[1 - \sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,z^k\right], \end{aligned}$$

where $k = 2j$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{2}{\pi}\,\frac{z^2}{z^2 - 1}\left[1 - \sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,z^k\right],$$

where $k = 2j$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z^2}{z^2 - 1} = \frac{1}{2} - \frac{\imath}{2}\,\frac{\cos(\theta)}{\sin(\theta)},$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{1}{\pi}\left[1 - \imath\,\frac{\cos(\theta)}{\sin(\theta)}\right]\left[1 - \sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,\cos(k\theta) - \imath\sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,\sin(k\theta)\right] \\ &= \frac{1}{\pi\sin(\theta)}\left\{\sin(\theta) - \sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,\sin[(k+1)\theta] + \right. \\ &\qquad \left. -\,\imath\cos(\theta) + \imath\sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,\cos[(k+1)\theta]\right\}, \end{aligned}$$

where $k = 2j$, and where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{1}{\pi\sin(\theta)}\left\{-\cos(\theta) + \sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,\cos[(k+1)\theta]\right\},$$

where $k = 2j$, and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta) = \frac{1}{\pi\sin(\theta)}\left\{\sin(\theta) - \sum_{j=1}^{\infty}\frac{4}{k(k+2)}\,\sin[(k+1)\theta]\right\},$$

where $k = 2j$.

A.3.2 Second-Order Center Series

There are two dominant singularities, located at $z = 1$ and at $z = -1$, so that we must use factors of $(z-1)^2(z+1)^2 = \left(z^2-1\right)^2$ in the construction of the second-order center series,

$$S_z = \frac{1}{\left(z^2-1\right)^2}\, C_z,$$

where

$$\begin{aligned} C_z &= -\frac{4}{\pi}\left(z^2-1\right)^2 \sum_{j=1}^{\infty} \frac{1}{k}\, z^k \\ &= -\frac{1}{\pi}\, z^2 \left[2 - 3z^2 + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)}\, z^{k+2} \right], \end{aligned}$$

where $k = 2j$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^3$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = -\frac{1}{\pi}\, \frac{z^2}{\left(z^2-1\right)^2} \left[2 - 3z^2 + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)}\, z^{k+2} \right],$$

where $k = 2j$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho \exp(\imath\theta)$ we have on the unit circle

$$\frac{z^2}{\left(z^2-1\right)^2} = -\frac{1}{4\sin^2(\theta)}.$$

We also have that

$$2 - 3z^2 = \left[-1 + 6\sin^2(\theta)\right] + \imath\left[-6\sin(\theta)\cos(\theta)\right],$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z = \frac{1}{4\pi\sin^2(\theta)} \Bigg\{ & -1 + 6\sin^2(\theta) + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)} \cos[(k+2)\theta] + \\ & -\imath 6\sin(\theta)\cos(\theta) + \imath \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)} \sin[(k+2)\theta] \Bigg\}, \end{aligned}$$

where $k = 2j$, and where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{1}{4\pi\sin^2(\theta)} \left\{ -6\sin(\theta)\cos(\theta) + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)} \sin[(k+2)\theta] \right\},$$

where $k = 2j$, and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta) = \frac{1}{4\pi\sin^2(\theta)} \left\{ -1 + 6\sin^2(\theta) + \sum_{j=1}^{\infty} \frac{32}{k(k+2)(k+4)} \cos[(k+2)\theta] \right\},$$

where $k = 2j$.

A.4 The Triangular Wave

Consider the unit-amplitude triangular wave, given by the cosine series

$$S_{\rm c} = -\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{k^2}\cos(k\theta),$$

where $k = 2j+1$. The corresponding FC series is then

$$S_{\rm s} = -\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{k^2}\sin(k\theta),$$

where $k = 2j+1$. Note that due to the factors of $1/k^2$ these series are already absolutely and uniformly convergent. The complex power series S_z is given by

$$S_z = -\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{k^2}z^k,$$

where $k = 2j+1$, of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.4.1 First-Order Center Series

There are two dominant singularities, located at $z = 1$ and at $z = -1$, so that we must use factors of $(z-1)(z+1) = z^2 - 1$ in the construction of the first-order center series,

$$S_z = \frac{1}{z^2-1}C_z,$$

where

$$\begin{aligned} C_z &= -\frac{8}{\pi^2}\left(z^2-1\right)\sum_{j=0}^{\infty}\frac{1}{k^2}z^k \\ &= \frac{8}{\pi^2}z\left[1-\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}z^{k+1}\right], \end{aligned}$$

where $k = 2j+1$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k^2$, this series has coefficients that go to zero as $1/k^3$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{8}{\pi^2}\frac{z}{z^2-1}\left[1-\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}z^{k+1}\right],$$

where $k = 2j+1$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{z^2-1} = -\frac{\imath}{2\sin(\theta)},$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= -\frac{4}{\pi^2}\,\frac{\imath}{\sin(\theta)}\left\{1-\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\,\cos[(k+1)\theta]-\imath\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\,\sin[(k+1)\theta]\right\} \\ &= \frac{4}{\pi^2\sin(\theta)}\left\{-\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\,\sin[(k+1)\theta]\,-\,\imath+\imath\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\,\cos[(k+1)\theta]\right\}, \end{aligned}$$

where $k = 2j + 1$, and where we collected the real and imaginary terms. The original DP function is given by the real part,

$$f_{\rm c}(\theta) = -\frac{4}{\pi^2\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\,\sin[(k+1)\theta]\right\},$$

where $k = 2j + 1$ and the corresponding FC function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{4}{\pi^2\sin(\theta)}\left\{-1+\sum_{j=0}^{\infty}\frac{4(k+1)}{k^2(k+2)^2}\,\cos[(k+1)\theta]\right\},$$

where $k = 2j + 1$.

A.5 The Shifted Square Wave

Consider the shifted unit-amplitude square wave, given by the cosine series

$$S_{\rm c} = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\,\cos(k\theta),$$

where $k = 2j + 1$. The corresponding FC series is then

$$S_{\rm s} = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\,\sin(k\theta),$$

where $k = 2j + 1$, and the complex power series S_z is given by

$$S_z = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\,z^k,$$

where $k = 2j + 1$, of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.5.1 First-Order Center Series

There are two dominant singularities, located at $z = \imath$ and at $z = -\imath$, so that we must use factors of $(z - \imath)(z + \imath) = z^2 + 1$ in the construction of the first-order center series,

$$S_z = \frac{1}{z^2+1}\,C_z,$$

where

$$
\begin{aligned}
C_z &= \frac{4}{\pi}\left(z^2+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\,z^k \\
&= \frac{4}{\pi}\,z\left[1+\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\,z^{k+1}\right],
\end{aligned}
$$

where $k = 2j + 1$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$
S_z = \frac{4}{\pi}\,\frac{z}{z^2+1}\left[1+\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\,z^{k+1}\right],
$$

where $k = 2j + 1$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$
\frac{z}{z^2+1} = \frac{1}{2\cos(\theta)},
$$

and therefore we have for S_z on the unit circle

$$
S_z = \frac{2}{\pi\cos(\theta)}\left\{1+\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\,\cos[(k+1)\theta] + \imath\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\,\sin[(k+1)\theta]\right\},
$$

where $k = 2j + 1$, and where we collected the real and imaginary terms. The original DP function is given by the real part,

$$
f_{\rm c}(\theta) = \frac{2}{\pi\cos(\theta)}\left\{1+\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\,\cos[(k+1)\theta]\right\},
$$

where $k = 2j + 1$, and the corresponding FC function is given by the imaginary part,

$$
f_{\rm s}(\theta) = \frac{2}{\pi\cos(\theta)}\left\{\sum_{j=0}^{\infty}\frac{2(-1)^j}{k(k+2)}\,\sin[(k+1)\theta]\right\},
$$

where $k = 2j + 1$.

A.5.2 Second-Order Center Series

There are two dominant singularities, located at $z = \imath$ and at $z = -\imath$, so that we must use factors of $(z-\imath)^2(z+\imath)^2 = \left(z^2+1\right)^2$ in the construction of the second-order center series,

$$
S_z = \frac{1}{\left(z^2+1\right)^2}\,C_z,
$$

where

$$\begin{aligned} C_z &= \frac{4}{\pi}\left(z^2+1\right)^2\sum_{j=0}^{\infty}\frac{(-1)^j}{k}\,z^k \\ &= \frac{4}{3\pi}\,z\left[3+5z^2+\sum_{j=0}^{\infty}\frac{24(-1)^j}{k(k+2)(k+4)}\,z^{k+3}\right], \end{aligned}$$

where $k = 2j + 1$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^3$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{4}{3\pi}\,\frac{z}{\left(z^2+1\right)^2}\left[3+5z^2+\sum_{j=0}^{\infty}\frac{24(-1)^j}{k(k+2)(k+4)}\,z^{k+3}\right],$$

where $k = 2j + 1$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{(z^2+1)} = \frac{z^*}{4\cos^2(\theta)},$$

where $z^*z = 1$. We also have that

$$\begin{aligned} z^*\left(3+5z^2\right) &= 3z^*+5z \\ &= [8\cos(\theta)]+\imath[2\sin(\theta)], \end{aligned}$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{2}{3\pi\cos^2(\theta)}\left\{4\cos(\theta)+\sum_{j=0}^{\infty}\frac{12(-1)^j}{k(k+2)(k+4)}\,\cos[(k+2)\theta]+\right. \\ &\qquad \left. +\imath\sin(\theta)+\imath\sum_{j=0}^{\infty}\frac{12(-1)^j}{k(k+2)(k+4)}\,\sin[(k+2)\theta]\right\}, \end{aligned}$$

where $k = 2j + 1$, and where we collected the real and imaginary terms. The original DP function is given by the real part,

$$f_{\rm c}(\theta) = \frac{2}{3\pi\cos^2(\theta)}\left\{4\cos(\theta)+\sum_{j=0}^{\infty}\frac{12(-1)^j}{k(k+2)(k+4)}\,\cos[(k+2)\theta]\right\},$$

where $k = 2j + 1$, and the corresponding FC function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{2}{3\pi\cos^2(\theta)}\left\{\sin(\theta)+\sum_{j=0}^{\infty}\frac{12(-1)^j}{k(k+2)(k+4)}\,\sin[(k+2)\theta]\right\},$$

where $k = 2j + 1$.

A.6 The Parabolic Wave

Consider a continuous and differentiable periodic function built with segments of quadratic functions, given by the sine series

$$S_{\rm s} = \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{1}{k^3} \sin(k\theta),$$

where $k = 2j + 1$. The corresponding FC series is then

$$S_{\rm c} = \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{1}{k^3} \cos(k\theta),$$

where $k = 2j + 1$. Note that due to the factors of $1/k^3$ these series are already absolutely and uniformly convergent. The complex power series S_z is given by

$$S_z = \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{1}{k^3} z^k,$$

where $k = 2j + 1$, of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.6.1 First-Order Center Series

There are two dominant singularities, located at $z = 1$ and at $z = -1$, so that we must use factors of $(z - 1)(z + 1) = z^2 - 1$ in the construction of the first-order center series,

$$S_z = \frac{1}{z^2 - 1} C_z,$$

where

$$\begin{aligned} C_z &= \frac{32}{\pi^3} \left(z^2 - 1\right) \sum_{j=0}^{\infty} \frac{1}{k^3} z^k \\ &= \frac{32}{\pi^3} z \left[-1 + \sum_{j=0}^{\infty} \frac{6k(k+2) + 8}{k^3(k+2)^3} z^{k+1} \right], \end{aligned}$$

where $k = 2j + 1$, and where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k^3$, this series has coefficients that go to zero as $1/k^4$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{32}{\pi^3} \frac{z}{z^2 - 1} \left[-1 + \sum_{j=0}^{\infty} \frac{6k(k+2) + 8}{k^3(k+2)^3} z^{k+1} \right],$$

where $k = 2j + 1$. In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho \exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{z^2 - 1} = -\frac{\imath}{2 \sin(\theta)},$$

and therefore we have for S_z on the unit circle

$$\begin{aligned}
S_z &= -\frac{16}{\pi^3}\frac{\imath}{\sin(\theta)}\times \\
&\times\left\{-1+\sum_{j=0}^{\infty}\frac{6k(k+2)+8}{k^3(k+2)^3}\cos[(k+1)\theta]+\imath\sum_{j=0}^{\infty}\frac{6k(k+2)+8}{k^3(k+2)^3}\sin[(k+1)\theta]\right\} \\
&= \frac{16}{\pi^3\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{6k(k+2)+8}{k^3(k+2)^3}\sin[(k+1)\theta]+\right. \\
&\qquad\left.+\imath-\imath\sum_{j=0}^{\infty}\frac{6k(k+2)+8}{k^3(k+2)^3}\cos[(k+1)\theta]\right\},
\end{aligned}$$

where $k = 2j + 1$, and where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{16}{\pi^3\sin(\theta)}\left\{1-\sum_{j=0}^{\infty}\frac{6k(k+2)+8}{k^3(k+2)^3}\cos[(k+1)\theta]\right\},$$

where $k = 2j + 1$, and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta) = \frac{16}{\pi^3\sin(\theta)}\left\{\sum_{j=0}^{\infty}\frac{6k(k+2)+8}{k^3(k+2)^3}\sin[(k+1)\theta]\right\},$$

where $k = 2j + 1$.

A.7 The First Slow-Converging Case

Consider the odd function defined by the very slowly convergent sine series

$$S_{\rm s} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\sin(k\theta).$$

The corresponding FC series is then

$$S_{\rm c} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\cos(k\theta),$$

and the complex power series S_z is given by

$$S_z = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}z^k,$$

of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.7.1 First-Order Center Series

There is a single dominant singularity at $z = 1$, so that we must use a single factor of $(z-1)$ in the construction of the first-order center series,

$$S_z = \frac{1}{z-1}\,C_z,$$

where

$$\begin{aligned} C_z &= \frac{2}{\pi}\,(z-1)\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\,z^k \\ &= \frac{2}{\pi}\,z\left[-1+\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,z^k\right], \end{aligned}$$

where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k^{(1/2)}$, this series has coefficients that go to zero as $1/k^{(3/2)}$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{2}{\pi}\,\frac{z}{z-1}\left[-1+\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,z^k\right].$$

In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{z-1} = \frac{1}{2} - \frac{\imath}{2}\,\frac{\sin(\theta)}{1-\cos(\theta)}.$$

If we write this in terms of $\theta/2$ we get

$$\frac{z}{z-1} = \frac{1}{2} - \frac{\imath}{2}\,\frac{\cos(\theta/2)}{\sin(\theta/2)},$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{1}{\pi}\left[1-\imath\,\frac{\cos(\theta/2)}{\sin(\theta/2)}\right]\left[-1+\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\cos(k\theta)+\right. \\ &\qquad\left.+\imath\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\sin(k\theta)\right] \\ &= \frac{1}{\pi\sin(\theta/2)}\left\{-\sin(\theta/2)+\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\sin[(k+1/2)\theta]\,+\right. \\ &\qquad\left.+\,\imath\cos(\theta/2)-\imath\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\cos[(k+1/2)\theta]\right\}, \end{aligned}$$

where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta) = \frac{1}{\pi\sin(\theta/2)}\left\{\cos(\theta/2)-\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\cos[(k+1/2)\theta]\right\},$$

and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta)=\frac{1}{\pi\sin(\theta/2)}\left\{-\sin(\theta/2)+\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}(k+1)+k\sqrt{k+1}}\,\sin[(k+1/2)\theta]\right\}.$$

A.7.2 Second-Order Center Series

There is a single dominant singularity at $z=1$, so that we must use factors of $(z-1)^2$ in the construction of the second-order center series,

$$S_z=\frac{1}{(z-1)^2}\,C_z,$$

where

$$\begin{aligned}C_z &= \frac{2}{\pi}\,(z-1)^2\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\,z^k\\ &= \frac{1}{\pi}\,z\left[2-\left(4-\sqrt{2}\right)z+\right.\\ &\qquad\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,z^{k+1}\right],\end{aligned}$$

where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k^{(1/2)}$, this series has coefficients that go to zero as $1/k^{(5/2)}$ when $k\to\infty$ (although this is not immediately obvious), and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$\begin{aligned}S_z &= \frac{1}{\pi}\,\frac{z}{(z-1)^2}\times\\ &\quad\times\left[2-\left(4-\sqrt{2}\right)z+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,z^{k+1}\right].\end{aligned}$$

In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z=\rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{(z-1)^2}=\frac{1}{2[\cos(\theta)-1]}.$$

If we write this in terms of $\theta/2$ we get

$$\frac{z}{(z-1)^2}=-\,\frac{1}{4\sin^2(\theta/2)}.$$

We also have that

$$\begin{aligned}2-\left(4-\sqrt{2}\right)z &= \left[2-\left(4-\sqrt{2}\right)\cos(\theta)\right]+\imath\left[-\left(4-\sqrt{2}\right)\sin(\theta)\right]\\ &= \left[-\left(2-\sqrt{2}\right)+\left(8-2\sqrt{2}\right)\sin^2(\theta/2)\right]+\\ &\quad+\imath\left[-\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)\right],\end{aligned}$$

and therefore we have for S_z on the unit circle

$$\begin{aligned}
S_z &= \frac{1}{4\pi\sin^2(\theta/2)}\times \\
&\times\left\{\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\sin^2(\theta/2)+\right. \\
&\quad-\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\cos[(k+1)\theta]+ \\
&\quad+\imath\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)+ \\
&\quad\left.-\imath\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\sin[(k+1)\theta]\right\},
\end{aligned}$$

where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$\begin{aligned}
f_{\rm s}(\theta) &= \frac{1}{4\pi\sin^2(\theta/2)}\times \\
&\times\left\{\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)+\right. \\
&\quad\left.-\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\sin[(k+1)\theta]\right\},
\end{aligned}$$

and the corresponding FC function is given by the real part,

$$\begin{aligned}
f_{\rm c}(\theta) &= \frac{1}{4\pi\sin^2(\theta/2)}\times \\
&\times\left\{\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\sin^2(\theta/2)+\right. \\
&\quad\left.-\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\cos[(k+1)\theta]\right\}.
\end{aligned}$$

A.8 The Second Slow-Converging Case

Consider the odd function defined by the very slowly convergent sine series

$$S_{\rm s}=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}\sin(k\theta).$$

The corresponding FC series is then

$$S_{\rm c}=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}\cos(k\theta),$$

and the complex power series S_z is given by

$$S_z=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}z^k,$$

of which the two DP Fourier series above are the real and imaginary parts on the unit circle.

A.8.1 First-Order Center Series

There is a single dominant singularity at $z = -1$, so that we must use a single factor of $(z + 1)$ in the construction of the first-order center series,

$$S_z = \frac{1}{z+1} C_z,$$

where

$$\begin{aligned} C_z &= \frac{2}{\pi}(z+1)\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}z^k \\ &= \frac{2}{\pi}z\left[-1+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}z^k\right], \end{aligned}$$

where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k^{(1/2)}$, this series has coefficients that go to zero as $1/k^{(3/2)}$ when $k \to \infty$, and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{2}{\pi}\frac{z}{z+1}\left[-1+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}z^k\right].$$

In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z = \rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{z+1} = \frac{1}{2} + \frac{\imath}{2}\frac{\sin(\theta)}{1+\cos(\theta)}.$$

If we write this in terms of $\theta/2$ we get

$$\frac{z}{z-1} = \frac{1}{2} + \frac{\imath}{2}\frac{\sin(\theta/2)}{\cos(\theta/2)},$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{1}{\pi}\left[1+\imath\frac{\sin(\theta/2)}{\cos(\theta/2)}\right]\left[-1+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\cos(k\theta)+\right. \\ &\qquad\left.+\imath\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\sin(k\theta)\right] \\ &= \frac{1}{\pi\cos(\theta/2)}\left\{-\cos(\theta/2)+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\cos[(k+1/2)\theta]+\right. \\ &\qquad\left.-\imath\sin(\theta/2)+\imath\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\sin[(k+1/2)\theta]\right\}, \end{aligned}$$

where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$f_s(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{-\sin(\theta/2)+\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\sin[(k+1/2)\theta]\right\},$$

and the corresponding FC function is given by the real part,

$$f_{\rm c}(\theta) = \frac{1}{\pi\cos(\theta/2)}\left\{-\cos(\theta/2) + \sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(k+1)+k\sqrt{k+1}}\cos[(k+1/2)\theta]\right\}.$$

A.8.2 Second-Order Center Series

There is a single dominant singularity at $z=-1$, so that we must use factors of $(z+1)^2$ in the construction of the second-order center series,

$$S_z = \frac{1}{(z+1)^2}\,C_z,$$

where

$$\begin{aligned} C_z &= \frac{2}{\pi}(z+1)^2\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}}\,z^k \\ &= \frac{1}{\pi}\,z\left[-2-\left(4-\sqrt{2}\right)z+\right. \\ &\quad\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}(-1)^k z^{k+1}\right], \end{aligned}$$

where we distributed the factor on the series and manipulated the indices of the resulting sums. Unlike the original series, with coefficients that behave as $1/k^{(1/2)}$, this series has coefficients that go to zero as $1/k^{(5/2)}$ when $k\to\infty$ (although this is not immediately obvious), and therefore our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$\begin{aligned} S_z &= \frac{1}{\pi}\,\frac{z}{(z+1)^2}\left[-2-\left(4-\sqrt{2}\right)z+\right. \\ &\quad\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}(-1)^k\,z^{k+1}\right]. \end{aligned}$$

In order to take the real and imaginary parts of S_z on the unit circle, we observe now that since $z=\rho\exp(\imath\theta)$ we have on the unit circle

$$\frac{z}{(z+1)^2} = \frac{1}{2[1+\cos(\theta)]}.$$

If we write this in terms of $\theta/2$ we get

$$\frac{z}{(z+1)^2} = \frac{1}{4\cos^2(\theta/2)}.$$

We also have that

$$\begin{aligned} -2-\left(4-\sqrt{2}\right)z &= \left[-2-\left(4-\sqrt{2}\right)\cos(\theta)\right]+\imath\left[-\left(4-\sqrt{2}\right)\sin(\theta)\right] \\ &= \left[\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\cos^2(\theta/2)\right]+ \\ &\quad+\imath\left[-\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)\right], \end{aligned}$$

and therefore we have for S_z on the unit circle

$$\begin{aligned} S_z &= \frac{1}{4\pi\cos^2(\theta/2)}\times \\ &\times\left\{\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\cos^2(\theta/2)+\right. \\ &\quad+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,(-1)^k\,\cos[(k+1)\theta]+ \\ &\quad-\boldsymbol{\imath}\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)+ \\ &\quad\left.+\boldsymbol{\imath}\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,(-1)^k\,\sin[(k+1)\theta]\right\}, \end{aligned}$$

where we collected the real and imaginary terms. The original DP function is given by the imaginary part,

$$\begin{aligned} f_{\rm s}(\theta) &= \frac{1}{4\pi\cos^2(\theta/2)}\times \\ &\times\left\{-\left(8-2\sqrt{2}\right)\sin(\theta/2)\cos(\theta/2)+\right. \\ &\quad\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,(-1)^k\sin[(k+1)\theta]\right\}, \end{aligned}$$

and the corresponding FC function is given by the real part,

$$\begin{aligned} f_{\rm c}(\theta) &= \frac{1}{4\pi\cos^2(\theta/2)}\times \\ &\times\left\{\left(2-\sqrt{2}\right)-\left(8-2\sqrt{2}\right)\cos^2(\theta/2)+\right. \\ &\quad\left.+\sum_{k=1}^{\infty}2\,\frac{\sqrt{k+1}\sqrt{k+2}-2\sqrt{k}\sqrt{k+2}+\sqrt{k}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}\sqrt{k+2}}\,(-1)^k\cos[(k+1)\theta]\right\}. \end{aligned}$$

B Numerical Results

As preliminary information, Table 2 gives the processing times of the runs used to generate the high-precision representation of the functions for which we do not have piece-wise expressions in closed form. This was done with either the first-order or the second-order center series, depending on the case. The table makes it clear that for all the functions tested a sufficiently high-order center series can be added up to high levels of precision in short times.

Function	**Order**	**Run Time**
Conjugate of the One-Cycle Sawtooth Wave	2^{nd}	35.35
Conjugate of the Standard Square Wave	2^{nd}	16.78
Conjugate of the Two-Cycle Sawtooth Wave	2^{nd}	16.89
Conjugate of the Triangular Wave	1^{st}	12.03
Conjugate of the Shifted Square Wave	2^{nd}	35.12
Conjugate of the Parabolic Wave	1^{st}	0.58
The First Slow-Converging Wave	2^{nd}	214.45
The Conjugate of the Function Above	2^{nd}	207.91
The Second Slow-Converging Wave	2^{nd}	412.91
The Conjugate of the Function Above	2^{nd}	404.63

Table 2: The processing time of each run used to produce the high-precision numerical representations of the functions, in seconds. The targeted precision level was 10^{-16}. The order of the center series used in each case is recorded.

The remaining tables can be found in the subsections that follow. They give, for each function tested, the number of added terms in each one of the two types of series, as a function of the required precision level, as well as the corresponding processing times. This is done for the average and the maximum numbers of added terms. The ratios shown represent the efficiency of the first-order center series as compared to the Fourier series. All processing times reported are for the calculation of the functions to the required precision at the complete set of valid points of the lattice within the interval $[-\pi, \pi]$.

The entries marked with crosses refer to the results from those runs that, as it turned out, it was not possible to execute within the limits of the available computational infrastructure. These are all runs using the Fourier series to compute the unlimited discontinuous functions. Derived numerical results that it was not possible to calculate are marked with N/A. In order to give some idea of the difficulties involved in the summation of the Fourier series, even for the relatively low precision levels involved, we may mention that the failed runs were interrupted while still incomplete after more than three months of processing time. According to the estimates that we are now able to work out, some of them would have gone up to thousands of years of processing time.

B.1 The One-Cycle Sawtooth Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	752.54	3252	35.60	1031	21.14	3.15
10^{-4}	7425.65	31432	115.41	2512	64.34	12.51
10^{-5}	73823.11	317182	367.13	7001	201.08	45.31
10^{-6}	737217.91	3174557	1161.60	21996	634.66	144.32
10^{-7}	7370774.30	31747807	3661.90	69494	2012.83	456.84
10^{-8}	73707293.40	317464682	11567.78	219994	6371.78	1443.06

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.10	0.01	10.00
10^{-4}	0.96	0.02	48.00
10^{-5}	9.92	0.05	198.40
10^{-6}	101.43	0.15	676.20
10^{-7}	1034.91	0.47	2201.94
10^{-8}	66297.97	1.48	44795.93

B.2 The Conjugate Function of the One-Cycle Sawtooth Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	752.56	3501	35.69	795	21.09	4.40
10^{-4}	7411.30	29995	114.99	2265	64.45	13.24
10^{-5}	73667.17	291994	366.73	7250	200.88	40.28
10^{-6}	735567.52	2870494	1160.84	21746	633.65	132.00
10^{-7}	7354439.12	28649994	3660.41	69244	2009.18	413.75
10^{-8}	73545041.88	286472994	11559.16	220244	6362.49	1300.71

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.09	0.00	N/A
10^{-4}	0.93	0.01	93.00
10^{-5}	9.60	0.04	240.00
10^{-6}	98.01	0.14	700.07
10^{-7}	1007.29	0.47	2143.17
10^{-8}	42957.88	1.45	29626.12

B.3 The Standard Square Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	752.71	3371	34.66	515	21.72	6.55
10^{-4}	7435.69	31270	113.36	1750	65.59	17.87
10^{-5}	74124.57	316870	360.47	5246	205.63	60.40
10^{-6}	740808.79	3179870	1134.01	15495	653.26	205.22
10^{-7}	7407676.63	31809120	3570.08	49244	2074.93	645.95
10^{-8}	74080415.47	318071620	11250.56	152494	6584.60	2085.80

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.05	0.00	N/A
10^{-4}	0.48	0.01	48.00
10^{-5}	4.78	0.04	119.5
10^{-6}	46.90	0.11	426.36
10^{-7}	720.70	0.32	2252.18
10^{-8}	19304.40	0.90	21449.33

B.4 The Conjugate Function of the Standard Square Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	752.57	3246	35.40	634	21.26	5.12
10^{-4}	7439.19	31495	113.26	1626	65.68	19.37
10^{-5}	74149.12	317245	360.66	5121	205.59	61.95
10^{-6}	741072.71	3179995	1134.23	15620	653.37	203.58
10^{-7}	7410279.77	31805495	3580.69	49119	2069.51	647.52
10^{-8}	74098529.06	317815995	11285.56	152369	6565.78	2085.83

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.06	0.00	N/A
10^{-4}	0.58	0.01	58.00
10^{-5}	5.75	0.03	191.67
10^{-6}	57.95	0.09	643.89
10^{-7}	924.20	0.30	3080.67
10^{-8}	23562.93	0.92	25611.88

B.5 The Two-Cycle Sawtooth Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	749.46	3121	34.02	515	22.03	6.06
10^{-4}	7419.26	31432	112.46	1750	65.97	17.96
10^{-5}	73816.53	317182	359.29	4996	205.45	63.49
10^{-6}	737570.08	3174557	1136.15	15744	649.18	201.64
10^{-7}	7375060.56	31747932	3582.81	48994	2058.46	648.00
10^{-8}	73753829.71	317472432	11324.11	157994	6512.99	2009.40

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.06	0.00	N/A
10^{-4}	0.53	0.01	53.00
10^{-5}	5.24	0.03	174.67
10^{-6}	51.51	0.09	572.33
10^{-7}	1504.69	0.28	5373.89
10^{-8}	57342.28	0.87	65910.67

B.6 The Conjugate Function of the Two-Cycle Sawtooth Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	748.73	3244	34.98	633	21.40	5.12
10^{-4}	7386.17	29994	112.46	1626	65.68	18.45
10^{-5}	73500.65	287744	357.86	4871	205.39	59.07
10^{-6}	734299.04	2864744	1135.19	15619	646.85	183.41
10^{-7}	7342346.76	28633244	3581.59	49119	2050.02	582.94
10^{-8}	73425099.64	286568244	11312.91	157869	6490.38	1815.23

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.06	0.00	N/A
10^{-4}	0.61	0.01	61.00
10^{-5}	5.92	0.02	296.00
10^{-6}	59.03	0.09	655.89
10^{-7}	988.73	0.28	3531.18
10^{-8}	26253.42	0.86	30527.23

B.7 The Triangular Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	12.69	52	6.67	102	1.90	0.51
10^{-4}	44.20	248	16.42	164	2.69	1.51
10^{-5}	142.62	508	37.51	390	3.80	1.30
10^{-6}	449.72	1006	82.16	641	5.47	1.57
10^{-7}	1417.96	3248	178.53	1135	7.94	2.86
10^{-8}	4469.11	9996	387.20	2626	11.54	3.81

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.00	0.00	N/A
10^{-4}	0.01	0.00	N/A
10^{-5}	0.01	0.01	1.00
10^{-6}	0.04	0.01	4.00
10^{-7}	0.12	0.02	6.00
10^{-8}	0.36	0.03	12.00

B.8 The Conjugate Function of the Triangular Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	13.07	91	6.58	63	1.99	1.44
10^{-4}	44.28	182	16.81	236	2.63	0.77
10^{-5}	141.83	393	37.31	286	3.80	1.37
10^{-6}	450.71	1129	82.31	525	5.48	2.15
10^{-7}	1420.28	3373	179.07	1257	7.93	2.68
10^{-8}	4477.51	9970	387.23	2749	11.56	3.63

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.00	0.00	N/A
10^{-4}	0.01	0.00	N/A
10^{-5}	0.02	0.00	N/A
10^{-6}	0.04	0.01	4.00
10^{-7}	0.11	0.02	5.50
10^{-8}	0.35	0.03	11.67

B.9 The Shifted Square Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	752.28	3371	34.64	515	21.72	6.55
10^{-4}	7431.44	31270	113.29	1750	65.60	17.87
10^{-5}	74082.23	316870	360.24	5246	205.65	60.40
10^{-6}	740385.81	3179870	1133.28	15495	653.31	205.22
10^{-7}	7403443.52	31809120	3567.77	49244	2075.09	645.95
10^{-8}	74034893.21	318100120	11243.29	152494	6584.81	2085.98

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.10	0.00	N/A
10^{-4}	0.94	0.02	47.00
10^{-5}	9.72	0.04	243.00
10^{-6}	98.73	0.14	705.21
10^{-7}	1016.27	0.45	2258.38
10^{-8}	36212.27	1.45	24973.98

B.10 The Conjugate Function of the Shifted Square Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	751.81	3246	35.36	634	21.26	5.12
10^{-4}	7431.74	31495	113.15	1626	65.68	9.37
10^{-5}	74074.90	317245	360.30	5121	205.59	61.95
10^{-6}	740331.15	3179995	1133.10	15620	653.37	203.58
10^{-7}	7402853.82	31804995	3577.10	49119	2069.51	647.51
10^{-8}	74022964.40	317512995	11274.26	152369	6565.66	2083.84

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.10	0.00	N/A
10^{-4}	0.97	0.02	48.5
10^{-5}	9.83	0.04	245.75
10^{-6}	100.74	0.15	671.6
10^{-7}	1029.30	0.48	2144.38
10^{-8}	44713.05	1.56	28662.21

B.11 The Parabolic Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	3.47	7	2.79	20	1.24	0.35
10^{-4}	9.12	22	6.31	40	1.45	0.55
10^{-5}	21.63	70	12.85	68	1.68	1.03
10^{-6}	48.51	145	24.33	126	1.99	1.15
10^{-7}	105.89	215	45.51	278	2.33	1.15
10^{-8}	229.68	455	82.33	511	2.79	0.89

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.00	0.00	N/A
10^{-4}	0.00	0.00	N/A
10^{-5}	0.00	0.00	N/A
10^{-6}	0.00	0.00	N/A
10^{-7}	0.00	0.00	N/A
10^{-8}	0.02	0.01	2.00

B.12 The Conjugate Function of the Parabolic Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	3.39	7	2.80	13	1.21	0.54
10^{-4}	9.05	22	6.43	39	1.41	0.56
10^{-5}	21.58	59	12.91	94	1.67	0.63
10^{-6}	48.36	126	24.56	157	1.97	0.80
10^{-7}	106.62	275	45.11	199	2.36	1.38
10^{-8}	229.69	518	82.02	409	2.80	1.27

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	0.00	0.00	N/A
10^{-4}	0.00	0.00	N/A
10^{-5}	0.00	0.00	N/A
10^{-6}	0.00	0.00	N/A
10^{-7}	0.01	0.00	N/A
10^{-8}	0.02	0.01	2.00

B.13 The First Slow-Converging Wave

The Number of Series Terms Added in Each Run						
Prec. Req.	Fourier Series		Center Series		Ratios FS/CS	
	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	1043658.86	10079182	99.74	4003	10463.79	2517.91
10^{-4}	104350817.73	1007942557	470.47	18496	221801.22	54495.16
10^{-5}	×.×	××	2173.32	80994	N/A	N/A
10^{-6}	×.×	××	10114.89	395494	N/A	N/A
10^{-7}	×.×	××	46909.65	1852994	N/A	N/A
10^{-8}	×.×	××	217611.91	8614994	N/A	N/A

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	84.17	0.01	8417.00
10^{-4}	20168.14	0.03	672271.33
10^{-5}	×.×	0.15	N/A
10^{-6}	×.×	0.68	N/A
10^{-7}	×.×	3.19	N/A
10^{-8}	×.×	14.78	N/A

B.14 The Conjugate Function of the First Slow-Converging Wave

The Number of Series Terms Added in Each Run						
Prec. Req.	Fourier Series		Center Series		Ratios FS/CS	
	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	1031902.95	8210494	100.73	4252	10244.25	1930.97
10^{-4}	103176107.38	820489994	472.20	18745	218500.86	43771.14
10^{-5}	10317645989.24	82058831494	2169.91	80744	4754872.78	1016283.95
10^{-6}	×.×	××	10097.77	395744	N/A	N/A
10^{-7}	×.×	××	46822.72	1845744	N/A	N/A
10^{-8}	×.×	××	216628.40	8294744	N/A	N/A

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	80.61	0.00	N/A
10^{-4}	14798.04	0.03	493268.00
10^{-5}	5679302.35	0.16	35495639.69
10^{-6}	×.×	0.79	N/A
10^{-7}	×.×	3.65	N/A
10^{-8}	×.×	16.81	N/A

B.15 The Second Slow-Converging Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	1044703.57	10079182	99.84	4003	10463.78	2517.91
10^{-4}	104455362.73	1007940557	470.94	18496	221801.85	54495.06
10^{-5}	10447231331.32	100761741932	2175.50	80994	4802220.79	1244064.28
10^{-6}	×.×	××	10125.02	395494	N/A	N/A
10^{-7}	×.×	××	46956.60	1852994	N/A	N/A
10^{-8}	×.×	××	217829.99	8614994	N/A	N/A

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	146.56	0.01	14656.00
10^{-4}	154870.85	0.06	2581180.83
10^{-5}	5394586.00	0.28	19266378.57
10^{-6}	×.×	1.34	N/A
10^{-7}	×.×	6.43	N/A
10^{-8}	×.×	29.95	N/A

B.16 The Conjugate Function of the Second Slow-Converging Wave

The Number of Series Terms Added in Each Run						
Prec.	Fourier Series		Center Series		Ratios FS/CS	
Req.	Average	Maximum	Average	Maximum	Average	Maximum
10^{-3}	1032834.30	8210494	100.82	4252	10244.34	1930.97
10^{-4}	103269333.48	820494994	472.59	18745	218517.81	43771.41
10^{-5}	10327775337.91	82168115994	2171.68	80744	4755661.67	1017637.42
10^{-6}	×.×	××	10106.03	395744	N/A	N/A
10^{-7}	×.×	××	46860.99	1845744	N/A	N/A
10^{-8}	×.×	××	216805.35	8294744	N/A	N/A

The Time of Each Run in Seconds			
Precision Required	Fourier Series Time	Center Series Time	Ratios FS/CS
10^{-3}	139.61	0.01	13961.00
10^{-4}	63223.87	0.06	1053731.17
10^{-5}	4041689.70	0.29	13936861.03
10^{-6}	×.×	1.34	N/A
10^{-7}	×.×	6.33	N/A
10^{-8}	×.×	29.82	N/A

References

[1] J. L. deLyra, "Fourier Theory on the Complex Plane I – Conjugate Pairs of Fourier Series and Inner Analytic Functions", arXiv: 1409.2582.

[2] J. L. deLyra, "Fourier Theory on the Complex Plane II – Weak Convergence, Classification and Factorization of Singularities", arXiv: 1409.4435.

[3] A compressed tar file containing the program used to plot the graphs of the functions, those used to perform the numerical tests, and some associated utilities, can be found at the URL

`http://latt.if.usp.br/scientific-pages/ntocs/`