# Numerical Solution for the Mean-Field Critical Curves of the $SO(\mathfrak{N})$ Polynomial $\lambda\varphi^4$ Models

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One of the important quantities one can calculate using the $N = 1$ mean-field method for the polynomial models is the position of the critical curve within the parameter plane of each model. This is done in section 3 of chapter 2 of the book [1], for the case of the $O(1)$ model. The result can be extended to the $SO(\mathfrak{N})$ models, and in this general case the $N = 1$ mean-field result for the critical curve $\lambda(\alpha)$ of the $\lambda\varphi^4$ polynomial model is:

$$\frac{\mathfrak{N}}{2d} = \frac{\displaystyle\int_0^\infty \mathrm{d}\varphi\ \varphi^{\mathfrak{N}+1}\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}}{\displaystyle\int_0^\infty \mathrm{d}\varphi\ \varphi^{\mathfrak{N}-1}\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}}, \tag{1}$$

which determines $\lambda(\alpha)$ implicitly. Since it is not possible to solve this equation analytically in order to obtain $\lambda(\alpha)$ in explicit form, we will be concerned here with the numerical solution of this equation, so that we may plot the graph of this function. The integrals can be written in terms of the parabolic cylinder functions $\mathbf{D}_\nu$ [3],

$$\frac{\sqrt{2\lambda}}{2d} = \frac{\mathbf{D}_{-\left(\frac{\mathfrak{N}}{2}+1\right)}\left(\frac{2d+\alpha}{\sqrt{2\lambda}}\right)}{\mathbf{D}_{-\left(\frac{\mathfrak{N}}{2}\right)}\left(\frac{2d+\alpha}{\sqrt{2\lambda}}\right)}.$$

However, this does not help us to solve the equation numerically because these functions are not readily available in a computer implementation. Note that for even $\mathfrak{N} = 2p$ the integrals can be written as integrals on a variable $\chi = \varphi^2$,

$$\frac{\mathfrak{N}}{2d} = \frac{\displaystyle\int_0^\infty \mathrm{d}\chi\ \chi^{p}\ e^{-\left[(d+\alpha/2)\chi+(\lambda/4)\chi^2\right]}}{\displaystyle\int_0^\infty \mathrm{d}\chi\ \chi^{p-1}\ e^{-\left[(d+\alpha/2)\chi+(\lambda/4)\chi^2\right]}}.$$

Hence it is not surprising that for even $\mathfrak{N}$ they can be written in terms of the error function, which can in fact be done through the $\mathbf{D}_{-n}$ functions, with integer $n$. The $\mathbf{D}_{-n}$ functions can be written in terms of the error function $\Phi(x)$, which is given by [4]

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\, e^{-t^2}.$$

For the first few functions $\mathbf{D}_{-n}$ we have [5]

$$\begin{aligned}
\mathbf{D}_0(x) &= e^{-x^2/4} \\
\mathbf{D}_{-1}(x) &= \sqrt{\frac{\pi}{2}}\, e^{x^2/4} \left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right] \\
\mathbf{D}_{-2}(x) &= \sqrt{\frac{\pi}{2}}\, e^{x^2/4} \left\{\sqrt{\frac{2}{\pi}}\, e^{-x^2/2} - x\left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right]\right\} \\
\mathbf{D}_{-3}(x) &= \frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{x^2/4} \left\{-\sqrt{\frac{2}{\pi}}\, x\, e^{-x^2/2} + (1+x^2)\left[1 - \Phi\left(\frac{x}{\sqrt{2}}\right)\right]\right\}
\end{aligned}$$

From the first two one can get all the functions in this sequence by repeated use of the recurrence relation [6]

$$\mathbf{D}_{p+1}(x) - x\mathbf{D}_p(x) + p\mathbf{D}_{p-1}(x) = 0,$$

which can also be written in the form

$$\mathbf{D}_{-(n+2)}(x) = \frac{\mathbf{D}_{-(n)}(x) - x\mathbf{D}_{-(n+1)}(x)}{n+1}.$$

Using all this one can solve the equation numerically for the case of even $\mathfrak{N}$, one case at a time, using the error functions `erf` or `derf`, which are readily available for use with the `g77` Fortran compiler. For example, for $\mathfrak{N} = 2$, which is the simplest case, the equation is

$$\sqrt{\frac{2}{\pi}}\, e^{-\frac{(2d+\alpha)^2}{4\lambda}} = \left(\frac{\sqrt{2\lambda}}{2d} + \frac{2d+\alpha}{\sqrt{2\lambda}}\right)\left[1 - \Phi\left(\frac{2d+\alpha}{2\sqrt{\lambda}}\right)\right].$$

Given a value of $\alpha$, one can solve this equation numerically for $\lambda$, and vice-versa. However, this solves only the even-$\mathfrak{N}$ cases, not the odd-$\mathfrak{N}$ cases, including $\mathfrak{N} = 1$.

For all values of $\mathfrak{N}$ one can make a change of variables in order to simplify the original integrals in a way that is convenient for numerical purposes. Since the solution for $\lambda = 0$ is obviously $\alpha = 0$, we can assume that $\lambda > 0$ and consider the variable $\chi$ given by

$$\chi = \left(\frac{\lambda}{4}\right)^{1/4} \varphi, \quad \varphi = \left(\frac{4}{\lambda}\right)^{1/4} \chi, \quad \mathrm{d}\varphi = \left(\frac{4}{\lambda}\right)^{1/4} \mathrm{d}\chi,$$

in terms of which the equation becomes

$$\frac{\mathfrak{N}}{2d} = \frac{2}{\sqrt{\lambda}} \frac{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}+1}\, e^{-\left[\frac{2d+\alpha}{\sqrt{\lambda}}\chi^2 + \chi^4\right]}}{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}-1}\, e^{-\left[\frac{2d+\alpha}{\sqrt{\lambda}}\chi^2 + \chi^4\right]}}.$$

Defining now a new parameter $\xi$ given by

$$\xi = -\frac{2d+\alpha}{2\sqrt{\lambda}} \Longrightarrow \alpha = -2\left(\xi\sqrt{\lambda} + d\right),$$

we may write the equation in the form

$$\sqrt{\lambda} = \frac{4d}{\mathfrak{N}} \frac{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}+1}\, e^{2\xi\chi^2 - \chi^4}}{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}-1}\, e^{2\xi\chi^2 - \chi^4}},$$

or, multiplying above and below by $\exp(-\xi^2)$,

$$\sqrt{\lambda} = \frac{4d}{\mathfrak{N}} \frac{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}+1}\, e^{-(\chi^2-\xi)^2}}{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}-1}\, e^{-(\chi^2-\xi)^2}}.$$

Given a value of $\xi$, this allows one to calculate $\sqrt{\lambda}$ using a single pair of numerical integrations, and afterward to calculate the corresponding $\alpha$ from $\xi$, $d$ and $\sqrt{\lambda}$. The parameter $\xi$ is a new parameter for the critical curve. In order to determine its range of variation as we travel along the curve from end to end, consider first the asymptotic line of the critical curve, for large values of $-\alpha$ and $\lambda$, which can be obtained through the representation of the equation in terms of the parabolic cylinder functions [7], and is given by

$$\lambda = -\frac{2d+\alpha}{\beta_c}, \qquad \beta_c = \frac{\mathfrak{N}}{2d},$$

where $\beta_c$ is the critical point of the sigma model over the arc at infinity. In this case we have for $\xi$ the expression

$$\xi = -\frac{2d+\alpha}{2\sqrt{\lambda}} = \frac{\beta_c\lambda}{2\sqrt{\lambda}} = \frac{1}{2}\beta_c\sqrt{\lambda}.$$

Therefore $\xi \to \infty$ corresponds to $\lambda \to \infty$ and hence to the sigma-model critical point over the arc at infinity, and when

$$\lambda \to \infty \quad \text{we have} \quad \xi \to \infty \quad \text{as} \quad \sqrt{\lambda}.$$

If we now consider the tangent to the critical curve at the Gaussian point, which can be obtained by differentiating equation (1) implicitly with respect to $\alpha$ and $\lambda$, then expressing the resulting integrals in terms of the $\Gamma$ function [8], and which is given by

$$\lambda = -\frac{\alpha}{\beta_c}, \qquad \beta_c = \frac{\mathfrak{N}}{2d},$$

we see that in this case we get for $\xi$ the expression

$$\xi = -\frac{2d+\alpha}{2\sqrt{\lambda}} = -\frac{d}{\sqrt{\lambda}} - \frac{\alpha}{2\sqrt{\lambda}} = -\frac{d}{\sqrt{\lambda}} + \frac{\beta_c\lambda}{2\sqrt{\lambda}} = -\frac{d}{\sqrt{\lambda}} + \frac{1}{2}\beta_c\sqrt{\lambda}.$$

Therefore $\xi \to -\infty$ corresponds to $\lambda \to 0$ and hence to the Gaussian point, and when

$$\lambda \to 0 \quad \text{we have} \quad \xi \to -\infty \quad \text{as} \quad \frac{1}{\sqrt{\lambda}}.$$

In short, we have the following limiting behaviors:

$$\begin{array}{rcccl} -\alpha \text{ and } \lambda & \to & \infty & \Longrightarrow & \xi \approx \frac{\mathfrak{N}}{4d}\sqrt{\lambda} \quad \to \quad \infty, \\ -\alpha \text{ and } \lambda & \to & 0 & \Longrightarrow & \xi \approx -\dfrac{d}{\sqrt{\lambda}} \quad \to \quad -\infty. \end{array}$$

We have therefore the following general algorithm for the numerical solution of the equation:

1. Choose a value for $\xi$ in $(-\infty, \infty)$.

2. Calculate $\sqrt{\lambda}$ by numerical integration,

$$\sqrt{\lambda} = \frac{4d}{\mathfrak{N}} \frac{\displaystyle\int_0^\infty d\chi\, \chi^{\mathfrak{N}+1}\, e^{-(\chi^2-\xi)^2}}{\displaystyle\int_0^\infty d\chi\, \chi^{\mathfrak{N}-1}\, e^{-(\chi^2-\xi)^2}}.$$

3. Calculate the corresponding $\alpha$ by the formula

$$\alpha = -2\left(\xi\sqrt{\lambda} + d\right).$$

The resulting pair $(\alpha, \lambda)$ is a point of the critical curve of the $SO(\mathfrak{N})$ model, in $d$ dimensions. Probably the best way to choose values of $\xi$ so that the resulting points $(\alpha, \lambda)$ are distributed fairly homogeneously along the critical curve is to use the equation of the tangent line at the Gaussian point, choosing some $\varepsilon$, integer values of $n = 1, 2, 3, \ldots$, and using

$$\xi_n = \frac{\mathfrak{N}}{4d}\sqrt{n\varepsilon} - \frac{d}{\sqrt{n\varepsilon}}.$$

Since the functions being integrated are centered around a single maximum that moves to large values of the variable when $\xi$ is large, in order to integrate them efficiently we must know the location of the points of maximum of the functions. The basic idea for the integration of one of these functions is to start at the point of maximum of the function, and then to integrate to both sides, down towards zero and up towards infinity. Let us consider then the functions

$$\chi^p\, e^{-(\chi^2-\xi)^2},$$

where $p \geq 0$ is an integer. Except for the case $p = 0$ this is zero for $\chi = 0$, and in all cases for $\chi \to \infty$. Taking the derivative we get

$$\chi^{p-1}[p + 4\xi\chi^2 - 4\chi^4]\, e^{-(\chi^2-\xi)^2}.$$

Besides vanishing at 0 (for $p > 1$) and at $\infty$, which are clearly points of minimum, this can be zero for

$$p + 4\xi\chi^2 - 4\chi^4 = 0 \Longrightarrow \chi^2 = \frac{\xi \pm \sqrt{\xi^2 + p}}{2},$$

and since $\chi^2 \geq 0$ only the + sign can be kept, and we get

$$\chi_{\max} = \sqrt{\frac{\xi + \sqrt{\xi^2 + p}}{2}}.$$

This is correct in all cases including $p = 0$. For $p = 0$ we have that, if $\xi > 0$, then $\chi_{\max} = \sqrt{\xi}$, and if $\xi \leq 0$, then $\chi_{\max} = 0$. Note that the only case in which the maximum is at zero is $p = 0$ with $\xi \leq 0$, in all other cases the maximum is at a positive and non-vanishing value of $\chi$.

The functions examined above have the form of a pulse around their points of maximum, but this pulse has a variable width, which decreases as the parameter $\xi$ increases to positive values. This makes is harder to find an appropriate numerical integration interval for them. For the cases $p > 0$ we can perform one more transformation of the integrals, in order to improve this situation. We make a transformation of integration variables to the variable $\gamma = \chi^2$ and hence write

$$\int_0^\infty \mathrm{d}\chi\, \chi^p\, e^{-(\chi^2-\xi)^2} = \frac{1}{2}\int_0^\infty \mathrm{d}\gamma\, \gamma^{\frac{p-1}{2}}\, e^{-(\gamma-\xi)^2},$$

where the integrand has now a fairly constant width around its maximum. In this case the location of the point of maximum is a bit different, that is, $\gamma_{\max}$ is not exactly $\chi^2_{\max}$. The integrand is now

$$\gamma^{\frac{p-1}{2}}\, e^{-(\gamma-\xi)^2},$$

and its derivative is given by

$$\left[\frac{p-1}{2}\gamma^{\frac{p-3}{2}} - 2(\gamma-\xi)\gamma^{\frac{p-1}{2}}\right] e^{-(\gamma-\xi)^2}.$$

Note that, due to the possibly negative powers of $\gamma$, the cases $p = 1$ and $p = 2$ should be examined separately. For $p = 1$ we have

$$-2(\gamma-\xi)e^{-(\gamma-\xi)^2},$$

which is zero at the point of minimum at infinity and at $\gamma = \xi$, which is therefore the point of maximum. Since $\gamma \geq 0$ this is valid only for $\xi \geq 0$, for $\xi < 0$ the function is monotonically decreasing with its maximum at $\gamma = 0$. For $p = 2$ we have

$$\left[\frac{1}{2}\gamma^{-\frac{1}{2}} - 2(\gamma-\xi)\gamma^{\frac{1}{2}}\right] e^{-(\gamma-\xi)^2}.$$

Note that in this case the function has an infinite derivative at the point $\gamma = 0$, which makes it unsuitable for numerical integration. Its point of maximum can be obtained as one of the roots of a quadratic polynomial and is in fact included in the general case, which we proceed to analyze. If we multiply it by $\gamma^{3/2}$, the derivative of the general case becomes

$$\gamma^{\frac{p}{2}}\left[\frac{p-1}{2} - 2(\gamma-\xi)\gamma\right] e^{-(\gamma-\xi)^2}.$$

Except for the cases $p = 1$ and $p = 2$ this vanishes at the points of minimum at $\gamma = 0$ and at $\gamma \to \infty$, and for all cases it vanishes at the point of maximum given by

$$\frac{p-1}{4} + \xi\gamma - \gamma^2 = 0 \Longrightarrow \gamma = \frac{\xi \pm \sqrt{\xi^2 + (p-1)}}{2},$$

where since $\gamma \geq 0$ only the $+$ sign can be kept, and we get

$$\gamma_{\text{max}} = \frac{\xi + \sqrt{\xi^2 + (p-1)}}{2}.$$

This is in fact correct in all cases including $p = 1$ and $p = 2$. For $p = 1$ we see that this gives the correct answer: if $\xi > 0$, then $\gamma_{\text{max}} = \xi$, and if $\xi \leq 0$, then $\gamma_{\text{max}} = 0$. Note that the only case in which the maximum is at zero is $p = 1$ with $\xi \leq 0$, in all other cases the maximum is at a positive and non-vanishing value of $\gamma$.

# Complete Algorithm

Let us record here the complete algorithm, with all the details. Recall that the integrals in the $\gamma$ variable should not be used in the cases $p = 0$ and $p = 2$, in which cases we should use the integrals in terms of the variable $\chi$. For all other values of $p$ we should use the integrals in terms of the variable $\gamma$ for ease of numerical integration.

1. Choose a value for $\xi$ in $(-\infty, \infty)$. The method for this is: choose some typically small $\varepsilon$ and integers $n = 1, 2, 3, \ldots$, then use the values

$$\xi_n = \frac{\mathfrak{N}}{4d}\sqrt{n\varepsilon} - \frac{d}{\sqrt{n\varepsilon}},$$

   where $n\varepsilon$ is a very rough estimate of the corresponding value of $\lambda$. For example, for $\mathfrak{N} = 1$ using $\varepsilon = 0.1$ and $n$ up to 1000 results on $\lambda$ up to approximately 40, and using $\varepsilon = 1.2$ and $n$ up to 1000 results on $\lambda$ up to approximately 1000.

2. Calculate the corresponding value of $\sqrt{\lambda}$ by means of two numerical integrations, in one of the two forms,

$$\sqrt{\lambda} = \frac{4d}{\mathfrak{N}} \frac{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}+1}\, e^{-(\chi^2-\xi)^2}}{\displaystyle\int_0^\infty \mathrm{d}\chi\, \chi^{\mathfrak{N}-1}\, e^{-(\chi^2-\xi)^2}} = \frac{4d}{\mathfrak{N}} \frac{\displaystyle\int_0^\infty \mathrm{d}\gamma\, \gamma^{\frac{\mathfrak{N}}{2}}\, e^{-(\gamma-\xi)^2}}{\displaystyle\int_0^\infty \mathrm{d}\gamma\, \gamma^{\frac{\mathfrak{N}-2}{2}}\, e^{-(\gamma-\xi)^2}},$$

   where the integrals in $\chi$ and in $\gamma$ are related by

$$I_p = \int_0^\infty \mathrm{d}\chi\, \chi^p\, e^{-(\chi^2-\xi)^2} = \frac{1}{2}\int_0^\infty \mathrm{d}\gamma\, \gamma^{\frac{p-1}{2}}\, e^{-(\gamma-\xi)^2}.$$

   The form of the integral to be used depends on the value of $p$:

   $p = 0$:

$$I_0(\xi) = \int_0^\infty \mathrm{d}\chi\, e^{-(\chi^2-\xi)^2}, \qquad \chi_{\max} = \sqrt{\frac{\xi + |\xi|}{2}}.$$

   $p = 1$:

$$I_1(\xi) = \frac{1}{2}\int_0^\infty \mathrm{d}\gamma\, e^{-(\gamma-\xi)^2}, \qquad \gamma_{\max} = \frac{\xi + |\xi|}{2}.$$

   $p = 2$:

$$I_2(\xi) = \int_0^\infty \mathrm{d}\chi\, \chi^2\, e^{-(\chi^2-\xi)^2}, \qquad \chi_{\max} = \sqrt{\frac{\xi + \sqrt{\xi^2 + 2}}{2}}.$$

$p \geq 3$:

$$I_p(\xi) = \frac{1}{2} \int_0^\infty \mathrm{d}\gamma \, \gamma^{\frac{p-1}{2}} \, e^{-(\gamma-\xi)^2}, \qquad \gamma_{\rm max} = \frac{\xi + \sqrt{\xi^2 + (p-1)}}{2}.$$

The integration is to be by cubic splines, in which the integration increment $\Delta I$ for a function $y(x)$ with derivative $y'(x)$, from a point $(x_1, y_1)$ to a point $(x_2, y_2)$, is given by

$$\Delta I = \frac{1}{2}(y_1 + y_2)\Delta x - \frac{1}{12}\Delta y' (\Delta x)^2,$$

where $\Delta x = x_2 - x_1$, $\Delta y' = y_2' - y_1'$, and so on. The functions involved and their derivatives are

$$\begin{aligned}
f_p(\chi) &= \chi^p \, e^{-(\chi^2-\xi)^2}, \\
g_p(\gamma) &= \gamma^{\frac{p-1}{2}} \, e^{-(\gamma-\xi)^2}, \\
f_p'(\chi) &= \chi^{p-1}[p + 4\xi\chi^2 - 4\chi^4] \, e^{-(\chi^2-\xi)^2}, \\
g_p'(\gamma) &= \gamma^{\frac{p-3}{2}} \left[\frac{p-1}{2} + 2\xi\gamma - 2\gamma^2\right] e^{-(\gamma-\xi)^2}.
\end{aligned}$$

3. Calculate the corresponding value of $\alpha$ using the formula

$$\alpha = -2\left(\xi\sqrt{\lambda} + d\right).$$

This completes the algorithm. One should also recall that for even $\mathfrak{N}$ the relevant integrals, with odd $p$, can be written in terms of the error function. In particular, for $\mathfrak{N} = 2$ one can use

$$\begin{aligned}
I_1(\xi) &= \frac{\sqrt{\pi}}{4} \, [1 + \Phi(\xi)], \\
I_3(\xi) &= \frac{\sqrt{\pi}}{4} \, \xi \, [1 + \Phi(\xi)] + \frac{1}{4} \, e^{-\xi^2}.
\end{aligned}$$

This can be used to solve the problem in this case, probably in a more efficient way. For larger even values of $\mathfrak{N}$ one has to derive the corresponding relations for each case in order to do this.

# Fortran Routines

Here is a list of the available Fortran routines, with a short description of each one. All the code is fairly well documented with internal comments. The source code is freely available at the URL:

`http://latt.if.usp.br/scientific-pages/nsmfccsonpm/`

`ff_0.f`: this routine implements the function $f_0(\chi)$; it is implemented as a double-precision function.

`fg_1.f`: this routine implements the function $g_1(\gamma)$; it is implemented as a double-precision function.

`ff_2.f`: this routine implements the function $f_2(\chi)$; it is implemented as a double-precision function.

`fg_p.f`: this routine implements the function $g_p(\gamma)$, for $p \geq 3$; it is implemented as a double-precision function.

`df_0.f`: this routine implements the derivative $f'_0(\chi)$; it is implemented as a double-precision function.

`dg_1.f`: this routine implements the derivative $g'_1(\gamma)$; it is implemented as a double-precision function.

`df_2.f`: this routine implements the derivative $f'_2(\chi)$; it is implemented as a double-precision function.

`dg_p.f`: this routine implements the derivative $g'_p(\gamma)$, for $p \geq 3$; it is implemented as a double-precision function.

The integrations in the routines `aif_*.f` and `aig_*.f` are performed in two passes, the first one uses a guessed integration range, and is used to measure the actual integration range. The integration starts at the maximum of the function and proceeds down to either side until a boundary of the range is found, or until the integral no longer increases. The second pass performs the final integration using this measured range.

The integration routines use be default 10000 intervals for the final integration. Considering that the cubic splines algorithm is precise to order $(\Delta x)^2$, this should give results which are good within single-precision level. This is certainly enough for plotting graphs, but if there is any need one can increase the number of integration intervals in order to get results good to double-precision level.

Note that the integration routines will crash for sufficiently large negative values of $\xi$, because the exponentials within the functions they integrate will eventually underflow and cause a division by zero.

`pars.f`: parameters common to all the integration routines, namely the value of a standard number of integration intervals, and the constant 1/12 for the cubic splines integration formula.

`aif_0.f`: this routine implements the integral from 0 to $\infty$, as a function of $\xi$, of $f_0(\chi)$; it is implemented as a double-precision function.

`aig_1.f`: this routine implements the integral from 0 to $\infty$, as a function of $\xi$, of $g_1(\gamma)$; it is implemented as a double-precision function.

`aif_2.f`: this routine implements the integral from 0 to $\infty$, as a function of $\xi$, of $f_2(\chi)$; it is implemented as a double-precision function.

`aig_p.f`: this routine implements the integral from 0 to $\infty$, as a function of $\xi$, of $g_p(\gamma)$, for $p \geq 3$; it is implemented as a double-precision function.

The programs `crit_*_curve.f` are written to read the input data from the `stdin` channel, and are to be used as encoded in the makefiles `Makefile.O1` through `Makefile.SO4`. They read in $d$, $\varepsilon$ and $n_{\rm max}$, then they loop over the resulting values of $\xi$ in order to produce the plotting data.

`crit_o1_curve.f`: this program produces the data to plot the graphs of the critical curves of the $O(1)$ model in any dimension $d$. It accepts from `stdin` the values of $d$, $\varepsilon$ and $n_{\rm max}$, these last two for the calculation of $\xi_n$.

`crit_so2_curve.f`: this program produces the data to plot the graphs of the critical curves of the $SO(2)$ model in any dimension $d$. It accepts from `stdin` the values of $d$, $\varepsilon$ and $n_{\rm max}$, these last two for the calculation of $\xi_n$.

`crit_so3_curve.f`: this program produces the data to plot the graphs of the critical curves of the $SO(3)$ model in any dimension $d$. It accepts from `stdin` the values of $d$, $\varepsilon$ and $n_{\rm max}$, these last two for the calculation of $\xi_n$.

`crit_so4_curve.f`: this program produces the data to plot the graphs of the critical curves of the $SO(4)$ model in any dimension $d$. It accepts from `stdin` the values of $d$, $\varepsilon$ and $n_{\rm max}$, these last two for the calculation of $\xi_n$.

# References


[1] "Non-Linear Scalar Models – A Continued Exploration of Quantum Field Theory", http://latt.if.usp.br/books/nlsm/.

[2] Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series and Products*, 4.th ed., Academic Press, 1965.

[3] *Ibid.*[2], formula 3.462, equation 1, page 337.

[4] *Ibid.*[2], formula 8.250, equation 1, page 930.

[5] *Ibid.*[2], formula 9.254, equations 1 and 2 (Attention: in old editions, the sign of this second formula is wrong!), page 1067.

[6] *Ibid.*[2], formula 9.247, equation 1, page 1066.

[7] *Ibid.*[2], formula 9.246, equation 2, pages 1065-1066.

[8] *Ibid.*[2], formula 3.461, equations 2 and 3, page 337.