Fourier Theory on the Complex Plane V
Arbitrary-Parity Real Functions, Singular Generalized Functions and Locally Non-Integrable Functions

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Abstract

A previously established correspondence between definite-parity real functions and inner analytic functions is generalized to real functions without definite parity properties. The set of inner analytic functions that corresponds to the set of all integrable real functions is then extended to include a set of singular "generalized functions" by the side of the integrable real functions. A general definition of these generalized functions is proposed and explored. The generalized functions are introduced loosely in the spirit of the Schwartz theory of distributions, and include the Dirac delta "function" and its derivatives of all orders. The inner analytic functions corresponding to this infinite set of singular real objects are given by means of a recursion relation. The set of inner analytic functions is then further extended to include a certain class of non-integrable real functions. The concept of integral-differential chains is used to help to integrate both the normal functions and the singular generalized functions seamlessly into a single structure. It does the same for the class of non-integrable real functions just mentioned. This extended set of generalized functions also includes arbitrary real linear combinations of all these real objects. An interesting connection with the Dirichlet problem on the unit disk is established and explored.

1 Introduction

In a previous paper [1] we established a relation between Definite-Parity (DP) real functions $f(\theta)$ defined on the interval $[-\pi, \pi]$ and analytic functions $w(z)$ within the open unit disk of the complex plane, as well as between the Fourier series of the DP real functions and the complex power series of the corresponding analytic functions, which we named "inner analytic functions". We refer the reader to that paper for the detailed definition and discussion of many of the concepts and notations we will use here. The DP real functions are interpreted as the restrictions of these inner analytic functions to the unit circle of the complex plane, and the Fourier series of the DP real functions as the restrictions of the complex Taylor series of the corresponding inner analytic functions to that same circle.

Within this context we showed, in a subsequent paper [3], that every DP real function within $[-\pi, \pi]$ that is absolutely integrable is associated to a specific inner analytic function and can be recovered from it almost everywhere in the limit from within the open unit disk to the unit circle. This is true even if the corresponding Fourier series is divergent, and the recovery of the DP real function can be executed using only its sequence of Fourier coefficients a_k, which are thus seen to uniquely characterize the DP real function almost everywhere. Let us emphasize that, when we talk in this paper of the recovery of a real function as the limit of an inner analytic function to the unit circle, we always mean recovery almost everywhere, that is, except possibly in a zero-measure subset of the domain.

On the other hand, if we consider the set of all possible inner analytic functions, we realize that it includes much more that just those inner analytic functions which are associated to the integrable DP real functions. This is illustrated by the interesting and suggestive fact that a radically singular object such as the Dirac delta "function" can also be represented by an inner analytic function, as was shown in detail in [1]. As we will see, some non-integrable real functions can also be represented by inner analytic functions. This at once poses the question of what is the complete set of real objects on the unit circle that corresponds to the set of all inner analytic functions within the open unit disk.

Here we propose to define the objects within this set as *generalized functions*, loosely in the spirit of the Schwartz theory of distributions [4], which will include objects such as the delta "function", its derivatives of arbitrarily high orders, and possibly other singular objects. Although we will stop short of giving a complete and detailed characterization of all possible such generalized functions, we will show that many of the better-known ones are included in the set. In fact, the inner analytic functions that correspond to the delta "function" and to its derivatives of all orders will be exhibited by a process of finite induction, leading to a simple algebraic recursion relation. We will also show that many non-integrable functions are in the set as well.

In order to better focus the analysis it may make good sense to impose some limitations on the definition of the generalized functions. The most general definition, as described above, and including both normal functions and generalized functions under the heading of "generalized functions", would be the following:

The set of all generalized functions $f(\theta)$ with domain on the interval $[-\pi, \pi]$ is the set of the limits of the real and imaginary parts of the inner analytic functions $w(z)$ from within the open unit disk to the unit circle, whenever these limits exist at least almost everywhere.

In this definition all possible inner analytic functions, as they were defined in [1], are included, and one may as well extend the definition to all analytic functions within the open unit disk. The extra conditions defining an inner analytic function $w(z)$, which were introduced in [1], are that $w(0) = 0$ and that $w(z)$ reduces to a real function on the $(-1, 1)$ interval of the real axis. These two extra conditions were originally imposed mostly in order to simplify the analysis. The first one is equivalent to the requirement that the real functions be zero-average functions, which is a trivial limitation since the addition of constant functions to the zero-average real functions, thus lifting the limitation, is a trivial operation that has no significant impact on any of the main results. The second one is equivalent to the requirement that the real and imaginary parts of $w(z)$, as well as the corresponding real functions, have definite parity properties with respect to θ. Since any real function $f(\theta)$ defined on $[-\pi, \pi]$ can be written in a unique way as a sum of an even function and an odd function, we see that this requirement is not really a limitation, and is used just to simplify the analysis.

The separation into even and odd parts can be easily applied to the generalized functions as well as to normal functions. For example, the Dirac delta "function" for an arbitrary singular point θ_1 is neither zero-average nor definite-parity, but the following zero-average combination can be separated into a zero-average even part and an odd part,

$$\begin{aligned}
\delta(\theta-\theta_1)-\frac{1}{2\pi} &= \mathfrak{E}\left[\delta(\theta-\theta_1)-\frac{1}{2\pi}\right]+\mathfrak{O}\left[\delta(\theta-\theta_1)-\frac{1}{2\pi}\right], \\
\mathfrak{E}\left[\delta(\theta-\theta_1)-\frac{1}{2\pi}\right] &= \frac{\delta(\theta-\theta_1)+\delta(\theta+\theta_1)}{2}-\frac{1}{2\pi}, \\
\mathfrak{O}\left[\delta(\theta-\theta_1)-\frac{1}{2\pi}\right] &= \frac{\delta(\theta-\theta_1)-\delta(\theta+\theta_1)}{2},
\end{aligned} \tag{1}$$

where we employ the symbol $\mathfrak{E}$ for the operation of taking the even part and the symbol $\mathfrak{O}$ for the operation of taking the odd part, and where we used the fact that $\delta(-\alpha) = \delta(\alpha)$, that is, the fact that the delta "function" centered at zero is even. Therefore, by representing by means of inner analytic functions within the open unit disk the even and odd singular "functions" on the right-hand sides of the last two equations above, each with two singularities, located at θ_1 and at $-\theta_1$ on the unit circle, one may recover the representation of the delta "function" with a single arbitrary singular point θ_1.

One may restrict the general definition given above in a more significant way by restricting in another way the set of inner analytic functions to be considered. A restriction to a smaller set of inner analytic functions, which we will adopt here as sufficient for our current purposes, is to consider only the set of inner analytic functions whose sequences of Taylor-Fourier coefficients a_k, as defined in [1], do not increase with k as $k \to \infty$ faster than all powers of k. In more precise terms, we choose to restrict our set of inner analytic functions to those such that, given the sequence of Taylor-Fourier coefficients a_k, there is an integer $p > 0$ such that

$$\lim_{k\to\infty}\frac{a_k}{k^p}=0. \tag{2}$$

Note that, since the sequence of Fourier coefficients a_k of any integrable real function is necessarily limited, this restriction does not exclude any such real functions. This condition defines a definite set of inner analytic functions within the open unit disk, and hence a definite set of generalized functions on the unit circle. This is a sufficiently general set of generalized functions for our purposes here, since, as we will see, it includes the Dirac delta "function" and its derivatives of arbitrarily high orders, as well as a certain class of non-integrable real functions.

As was discussed in the previous papers [1] and [3] mentioned above, besides DP real functions and their corresponding inner analytic functions it may also be useful to consider rotated inner analytic functions. These are analytic functions that correspond to shifted real functions such as $f(\theta-\theta_1)$ and which reduce to a real function on a diameter of the unit disk forming an angle θ_1 with the real axis. They constitute a fairly simple generalization of the original structure. We will use this generalization when we discuss a set of singular generalized functions in Section 3. However, in order to prepare the ground for the discussion of generalized functions, we will first consider the complete and detailed generalization of the correspondence between real functions and complex analytic functions to all integrable real functions, regardless of any parity considerations. The case of the rotated inner analytic functions will then become a particular case of this larger generalization.

Let us conclude this introduction with a note about the concept of integrability of real functions. What we mean by integrability of real functions in this paper is integrability in the sense of Lebesgue, with the use of the usual Lebesgue measure. We will assume that all the real functions under discussion here are measurable in this Lebesgue measure, regardless of whether or not they are integrable on their whole domain. Therefore whenever we speak of real functions in this paper, it should be understood that we mean Lebesgue-measurable real functions. We will then use the following result from the theory of measure and integration: within the set of all Lebesgue-measurable real functions defined on a compact interval, the conditions of integrability and of absolute integrability are two equivalent conditions [5]. Therefore we will use the concepts of integrability and of absolute integrability interchangeably, as convenience requires. An alternative integrability condition over the real functions, that we will also use, is the requirement that they be integrable in all closed sub-intervals of $[-\pi, \pi]$, which we will refer to as the condition of "local integrability". It can be shown that in this context this condition is in fact equivalent to the other two conditions of integrability. A simple proof of this equivalence can be found in Appendix A.

2 The General Case for Integrable Real Functions

Let us discuss the generalization of our results for DP real functions, obtained in [1] and [3], to a more general set of real functions. Let it be understood that, when we talk of real functions in this section, we always mean integrable zero-average real functions, but not necessarily DP real functions. As was shown in [3], any integrable DP real function $f(\theta)$ is representable almost everywhere by its Fourier coefficients a_k and is associated to an inner analytic function $w(z)$, whose Taylor coefficients are a_k, and from which it can be recovered almost everywhere in the $\rho \to 1$ limit, where $z = \rho \exp(\imath\theta)$. This is so because, given an even integrable real function $f_{\rm c}(\theta)$ we have that

$$\begin{aligned} f_{\rm c}(\theta) &= \lim_{\rho \to 1} \Re[w(z)] \\ &= \lim_{\rho \to 1} \Re\big[f_{\rm c}(\rho, \theta) + \imath \bar{f}_{\rm c}(\rho, \theta)\big] \\ &= \lim_{\rho \to 1} f_{\rm c}(\rho, \theta), \end{aligned}$$

where $\bar{f}_{\rm c}(\rho, \theta)$ is the Fourier-Conjugate (FC) function to $f_{\rm c}(\rho, \theta)$, as defined in [1], and where the inner analytic function associated to $f_{\rm c}(\theta)$ is given by

$$w(z) = f_{\rm c}(\rho, \theta) + \imath \bar{f}_{\rm c}(\rho, \theta).$$

Similarly, given an odd integrable real function $f_{\rm s}(\theta)$ we have that

$$\begin{aligned} f_{\rm s}(\theta) &= \lim_{\rho \to 1} \Im[w(z)] \\ &= \lim_{\rho \to 1} \Im\big[\bar{f}_{\rm s}(\rho, \theta) + \imath f_{\rm s}(\rho, \theta)\big] \\ &= \lim_{\rho \to 1} f_{\rm s}(\rho, \theta), \end{aligned}$$

where $\bar{f}_{\rm s}(\rho, \theta)$ is the FC function to $f_{\rm s}(\rho, \theta)$ and where the inner analytic function associated to $f_{\rm s}(\theta)$ is given by

$$w(z) = \bar{f}_{\rm s}(\rho, \theta) + \imath f_{\rm s}(\rho, \theta).$$

It is immediately apparent that, if we add two DP real functions with the same parity, we must simply add the two corresponding inner analytic functions in order to get the inner analytic function associated to the sum. For example, given two even DP real functions

$f_{1,\mathrm{c}}(\theta)$ and $f_{2,\mathrm{c}}(\theta)$, corresponding respectively to the inner analytic functions $w_1(z)$ and $w_2(z)$, we have that

$$\begin{aligned} f_{\mathrm{c}}(\theta) &= f_{1,\mathrm{c}}(\theta) + f_{2,\mathrm{c}}(\theta) \\ &= \lim_{\rho\to 1} \Re[w_1(z) + w_2(z)]\,, \end{aligned}$$

so that the inner analytic function corresponding to the sum $f_{\mathrm{c}}(\theta)$ is simply the sum

$$w(z) = w_1(z) + w_2(z).$$

The same is true for the addition of two odd DP real functions $f_{1,\mathrm{s}}(\theta)$ and $f_{2,\mathrm{s}}(\theta)$, and the corresponding inner analytic functions $w_1(z)$ and $w_2(z)$, since in this case we have that

$$\begin{aligned} f_{\mathrm{s}}(\theta) &= f_{1,\mathrm{s}}(\theta) + f_{2,\mathrm{s}}(\theta) \\ &= \lim_{\rho\to 1} \Im[w_1(z) + w_2(z)]\,, \end{aligned}$$

so that once more we get for the inner analytic function associated to the sum $f_{\mathrm{s}}(\theta)$ simply the sum

$$w(z) = w_1(z) + w_2(z).$$

However, if we add two DP real functions with opposite parities, say $f_{1,\mathrm{c}}(\theta)$ and $f_{2,\mathrm{s}}(\theta)$, then the situation changes a little, since in this case we have

$$\begin{aligned} f_{1,\mathrm{c}}(\theta) &= \lim_{\rho\to 1} \Re[w_1(z)] \\ &= \lim_{\rho\to 1} \Re\big[f_{1,\mathrm{c}}(\rho,\theta) + \imath \bar{f}_{1,\mathrm{c}}(\rho,\theta)\big]\,, \\ f_{2,\mathrm{s}}(\theta) &= \lim_{\rho\to 1} \Im[w_2(z)] \\ &= \lim_{\rho\to 1} \Im\big[\bar{f}_{2,\mathrm{s}}(\rho,\theta) + \imath f_{2,\mathrm{s}}(\rho,\theta)\big]\,. \end{aligned}$$

In this case we can construct an inner analytic function $w(z)$ such that the sum $f(\theta)$ of the two real functions is obtained from the $\rho \to 1$ limit of the real part of $w(z)$ by making a complex linear combination of the two corresponding inner analytic functions,

$$\begin{aligned} f(\theta) &= f_{1,\mathrm{c}}(\theta) + f_{2,\mathrm{s}}(\theta) \\ &= \lim_{\rho\to 1} \Re[w_1(z) - \imath w_2(z)] \\ &= \lim_{\rho\to 1} \Re\big\{[f_{1,\mathrm{c}}(\rho,\theta) + f_{2,\mathrm{s}}(\rho,\theta)] + \imath\,\big[\bar{f}_{1,\mathrm{c}}(\rho,\theta) - \bar{f}_{2,\mathrm{s}}(\rho,\theta)\big]\big\} \\ &= \lim_{\rho\to 1} [f_{1,\mathrm{c}}(\rho,\theta) + f_{2,\mathrm{s}}(\rho,\theta)] \\ &= f_{\mathrm{c}}(\theta) + f_{\mathrm{s}}(\theta), \end{aligned}$$

where $f_{\mathrm{c}}(\theta) = f_{1,\mathrm{c}}(\theta)$ is the even part of $f(\theta)$ and $f_{\mathrm{s}}(\theta) = f_{2,\mathrm{s}}(\theta)$ is its odd part. Since any real function $f(\theta)$ can be separated into its unique even and odd parts, this gives us an inner analytic function $w(z)$ from the real part of which this arbitrary real function can be obtained in the $\rho \to 1$ limit. This inner analytic function is given by

$$w(z) = w_1(z) - \imath w_2(z). \tag{3}$$

This solution for the inner analytic function corresponding to $f(\theta)$ is not unique, in the sense that it is possible to define another one, from the imaginary part of which the real function can also be obtained in the $\rho \to 1$ limit,

$$\begin{aligned}
f(\theta) &= f_{1,\mathrm{c}}(\theta) + f_{2,\mathrm{s}}(\theta) \\
&= \lim_{\rho\to 1} \Im[\imath w_1(z) + w_2(z)] \\
&= \lim_{\rho\to 1} \Im\big\{\big[-\bar{f}_{1,\mathrm{c}}(\rho,\theta) + \bar{f}_{2,\mathrm{s}}(\rho,\theta)\big] + \imath\,[f_{1,\mathrm{c}}(\rho,\theta) + f_{2,\mathrm{s}}(\rho,\theta)]\big\} \\
&= \lim_{\rho\to 1} [f_{1,\mathrm{c}}(\rho,\theta) + f_{2,\mathrm{s}}(\rho,\theta)] \\
&= f_{\mathrm{c}}(\theta) + f_{\mathrm{s}}(\theta),
\end{aligned}$$

so that we have for this alternative inner analytic function

$$w'(z) = \imath w_1(z) + w_2(z).$$

However, this second solution need not concern us, since it is in fact proportional to the first one, for we can see that $w'(z) = \imath w(z)$. So long as we agree that the real functions are to be obtained as the $\rho \to 1$ limits of, say, the real parts of the inner analytic functions, there is a unique inner analytic function associated to each real function. This can be applied to the purely odd real functions as well, if we associate to them the inner analytic function $-\imath w(z)$, rather than $w(z)$ as we have been doing so far, since we have that

$$\Im[w(z)] = \Re[-\imath w(z)].$$

In short, given any real function $f(\theta)$, regardless of whether or not it has definite parity properties, we know how to build from it the unique inner analytic function that gives us back that real function as the $\rho \to 1$ limit of its real part: first we separate the real function $f(\theta)$ into its even and odd parts $f_{1,\mathrm{c}}(\theta)$ and $f_{2,\mathrm{s}}(\theta)$; then, we use the previous method of definition, as given in [1], to determine the corresponding old-style inner analytic functions $w_1(z)$ and $w_2(z)$; finally, we define the inner analytic function corresponding to $f(\theta)$ as

$$w(z) = w_1(z) - \imath w_2(z).$$

Let us now consider the inverse problem, that is, given an arbitrary real function $f(\theta)$ and its corresponding inner analytic function $w(z)$, defined according to our new criterion, the problem of how to obtain from it the inner analytic functions corresponding to the even and odd parts of the real function. We may assume that the inner analytic function is given as

$$w(z) = f(\rho,\theta) + \imath\bar{f}(\rho,\theta),$$

where $f(\rho,\theta)$ is harmonic and $\bar{f}(\rho,\theta)$ is the harmonic conjugate function to $f(\rho,\theta)$, and where we assume adherence to the criterion that the real function is to be recovered from the $\rho \to 1$ limit of the real part of the corresponding inner analytic function,

$$\begin{aligned}
f(\theta) &= \lim_{\rho\to 1} \Re[w(z)] \\
&= \lim_{\rho\to 1} \Re[f(\rho,\theta) + \imath\bar{f}(\rho,\theta)] \\
&= \lim_{\rho\to 1} f(\rho,\theta).
\end{aligned}$$

On the other hand, we assume that $f(\theta)$ is decomposed into its even and odd parts as

$$f(\theta) = f_{1,\mathrm{c}}(\theta) + f_{2,\mathrm{s}}(\theta),$$

where the two DP real functions $f_{1,\mathrm{c}}(\theta)$ and $f_{2,\mathrm{s}}(\theta)$ are associated to the two inner analytic functions $w_1(z)$ and $w_2(z)$,

$$\begin{aligned} f_{1,\mathrm{c}}(\theta) &= \lim_{\rho\to 1} \Re[w_1(z)], \\ f_{2,\mathrm{s}}(\theta) &= \lim_{\rho\to 1} \Re[w_2(z)], \end{aligned}$$

which are defined according to our new criterion,

$$\begin{aligned} w_1(z) &= f_{1,\mathrm{c}}(\rho,\theta) + \boldsymbol{\imath}\bar{f}_{1,\mathrm{c}}(\rho,\theta), \\ w_2(z) &= f_{2,\mathrm{s}}(\rho,\theta) - \boldsymbol{\imath}\bar{f}_{2,\mathrm{s}}(\rho,\theta). \end{aligned}$$

Since we have that $w(z) = w_1(z) + w_2(z)$ we now see that we have for the inner analytic function $w(z)$ corresponding to $f(\theta)$, written in terms of the real and imaginary parts of the inner analytic functions $w_1(z)$ and $w_2(z)$ corresponding respectively to the even and odd parts of $f(\theta)$,

$$\begin{aligned} \left[f(\rho,\theta) + \boldsymbol{\imath}\bar{f}(\rho,\theta)\right] &= \left[f_{1,\mathrm{c}}(\rho,\theta) + \boldsymbol{\imath}\bar{f}_{1,\mathrm{c}}(\rho,\theta)\right] + \left[f_{2,\mathrm{s}}(\rho,\theta) - \boldsymbol{\imath}\bar{f}_{2,\mathrm{s}}(\rho,\theta)\right] \\ &= \left[f_{1,\mathrm{c}}(\rho,\theta) + f_{2,\mathrm{s}}(\rho,\theta)\right] + \boldsymbol{\imath}\left[\bar{f}_{1,\mathrm{c}}(\rho,\theta) - \bar{f}_{2,\mathrm{s}}(\rho,\theta)\right], \end{aligned}$$

which implies that we have

$$\begin{aligned} f(\rho,\theta) &= f_{1,\mathrm{c}}(\rho,\theta) + f_{2,\mathrm{s}}(\rho,\theta), \\ \bar{f}(\rho,\theta) &= \bar{f}_{1,\mathrm{c}}(\rho,\theta) - \bar{f}_{2,\mathrm{s}}(\rho,\theta). \end{aligned}$$

These two relations among real functions in turn imply that the DP functions on the right-hand sides are the even and odd parts with respect to θ of the functions in the left-hand sides. Characterizing the functions by their parities, and remembering that Fourier Conjugation reverses parity, we have

$$\begin{aligned} f_{1,\mathrm{c}}(\rho,\theta) &= \mathfrak{E}[f(\rho,\theta)], \\ f_{2,\mathrm{s}}(\rho,\theta) &= \mathfrak{O}[f(\rho,\theta)], \\ \bar{f}_{1,\mathrm{c}}(\rho,\theta) &= \mathfrak{O}[\bar{f}(\rho,\theta)], \\ \bar{f}_{2,\mathrm{s}}(\rho,\theta) &= -\mathfrak{E}[\bar{f}(\rho,\theta)], \end{aligned}$$

where the symbols $\mathfrak{E}$ and $\mathfrak{O}$ relate to the parity with respect to θ. As a consequence, we have for the inner analytic functions associated to the even and odd parts of $f(\theta)$, respectively

$$\begin{aligned} w_1(z) &= f_{1,\mathrm{c}}(\rho,\theta) + \boldsymbol{\imath}\bar{f}_{1,\mathrm{c}}(\rho,\theta) \\ &= \mathfrak{E}[f(\rho,\theta)] + \boldsymbol{\imath}\mathfrak{O}[\bar{f}(\rho,\theta)], \\ w_2(z) &= f_{2,\mathrm{s}}(\rho,\theta) - \boldsymbol{\imath}\bar{f}_{2,\mathrm{s}}(\rho,\theta) \\ &= \mathfrak{O}[f(\rho,\theta)] + \boldsymbol{\imath}\mathfrak{E}[\bar{f}(\rho,\theta)], \end{aligned}$$

where we are still using the new standard criterion that the real functions are to be recovered from the $\rho \to 1$ limits of the real parts of the corresponding inner analytic functions. In conclusion, given the inner analytic function $w(z)$ corresponding to $f(\theta)$, we may write for the inner analytic function $w_1(z)$ corresponding to the even part of $f(\theta)$, and for the inner analytic function $w_2(z)$ corresponding to the odd part of $f(\theta)$,

$$\begin{aligned} w_1(z) &= \mathfrak{E}\{\Re[w(z)]\} + \boldsymbol{\imath}\,\mathfrak{O}\{\Im[w(z)]\}, \\ w_2(z) &= \mathfrak{O}\{\Re[w(z)]\} + \boldsymbol{\imath}\,\mathfrak{E}\{\Im[w(z)]\}, \end{aligned}$$

which are both, therefore, uniquely and completely determined. We have, therefore, a consistent way to associate arbitrary real functions with corresponding inner analytic functions, in such a way that each real function is recovered as the $\rho \to 1$ limit of the real part of the corresponding inner analytic function. Due to this, the sum of any pair of real functions is now related to the simple sum of the corresponding pair of inner analytic functions. In other words, if we adhere to this new standard way to relate the real functions and the inner analytic functions, then the set of inner analytic functions is seen to inherit from the set of real functions its character as a vector space with real scalars.

One interesting special case, which will be of much use to us here, is that of a rotated inner analytic function. In order to discuss it, let us take the case of an even real function $f(\theta)$, and the corresponding inner analytic function $w(z)$. Let us suppose that we generate from $f(\theta)$ another function by just shifting the variable θ by a real constant θ_1, in order to define the shifted function $f_1(\theta) = f(\theta - \theta_1)$. Clearly this modified function is no longer an even function of θ. It has now non-zero even and odd parts, and the results previously obtained in this section can all be applied to it. Let us determine how the inner analytic function $w(z)$ corresponding to $f(\theta)$ changes into a new analytic function $w_1(z)$ that corresponds to the shifted real function $f_1(\theta)$. Let us recall that we have

$$z = \rho\, e^{\boldsymbol{\imath}\theta},$$

which is the only place there θ appears in the construction of the inner analytic function $w(z)$, and which, upon the shift $\theta \to \theta - \theta_1$, changes to

$$z' = \rho\, e^{\boldsymbol{\imath}(\theta-\theta_1)}.$$

If we define the complex constant z_1 associated to θ_1 as a point on the unit circle,

$$z_1 = e^{\boldsymbol{\imath}\theta_1},$$

then we have

$$\begin{aligned} z' &= \rho\, e^{\boldsymbol{\imath}\theta}\, e^{-\boldsymbol{\imath}\theta_1} \\ &= \frac{z}{z_1}. \end{aligned}$$

It follows therefore that this shift in the real variable θ corresponds to the exchange of z for the ratio z/z_1 in the inner analytic function, so that we simply have

$$w_1(z) = w(z/z_1).$$

Since $w(z)$ reduces to a real function on the interval $(-1, 1)$ of the real axis, it follows that $w_1(z)$ reduces to a real function when z/z_1 is real and within that interval. This is realized when z and z_1 are collinear in the complex plane, which means that we have $\theta = \theta_1$ or $\theta = \pi + \theta_1$. Therefore, the inner analytic function $w_1(z)$ reduces to a real function on a diameter of the unit circle that makes an angle θ_1 with the real axis. Similar conclusions are reached when $f(\theta)$ is an odd function. In this case the modified inner analytic function $w_1(z)$ is still given by the formula above, but it now reduces to a purely imaginary function over that same diagonal of the unit circle.

2.1 Updating Concepts and Definitions

We see therefore that the correspondence between real functions and inner analytic functions can be generalized in a simple and straightforward way to the set of all possible integrable real functions, regardless of any parity considerations. Upon this generalization the new class of inner analytic functions $w(z)$ that arises is somewhat larger than the original one. These are all still analytic functions within the open unit disk, and they do all have the property that $w(0) = 0$, just like the original ones, since we did not lift here the limitation that the real functions be zero-average. However, they no longer reduce to real functions over the interval $(-1, 1)$ of the real axis, nor they do so, in general, over any diameter of the unit disk. Only if the given real function is even will the corresponding inner analytic function reduce to a real function on that interval of the real axis. On the other hand, if the given real function is odd, then the corresponding inner analytic function, if defined according to our new criterion, reduces to a purely imaginary function on that same interval. It should be noted that, although this new class of inner analytic functions is a generalization of the previous one, it is still far from including all possible analytic functions within the open unit disk.

We see therefore that we have been led to generalize the concept of an inner analytic function when we worked out the generalization of our previous structure to non-DP real functions. It follows that several other concepts and definitions are in need of revision as well. For example, it is no longer true that the coefficients of the complex Taylor series are real. Also, the set of Fourier coefficients of the real functions consist now of two sequences, one for the even part and another for the odd part of the real function. Therefore, we can no longer use the notation a_k for the Fourier coefficients as we did before, where a_k could be the name for either the Fourier coefficients α_k of the cosine series of the even part or the Fourier coefficients β_k of the sine series of the odd part, as the case may be. We have now two sequences of Taylor-Fourier coefficients associated to each real function, and we must revert to the traditional notation α_k and β_k. In fact, given the new form of the inner analytic functions, shown in Equation (3), it follows that the coefficients of the complex power series are now given by

$$c_k = \alpha_k - \imath\beta_k.$$

Since α_k and β_k are both essentially arbitrary sequences of independent real coefficients, it follows that c_k is a sequence of essentially arbitrary complex coefficients. However, if we restrict ourselves to normal integrable real functions on the unit circle, then these coefficients are not really completely arbitrary, since α_k and β_k must satisfy the condition that they can be obtained as the usual integrals involving the real functions. Due to this, they are both limited sequences of coefficients, and therefore so is c_k.

Another concept that is in need of revision is that of Fourier Conjugation. In the previous paper [1] we defined the concept of Fourier Conjugation by the simple interchange of cosines and sines in the Fourier series of the DP real functions. This definition must now be adapted to the new larger structure, where it will still be related, as it was before, to the concept of harmonic conjugation. Let us state here our main new definitions.

Inner analytic function: an inner analytic function $w(z)$ is a complex function that is analytic within the open unit disk and that has the property that $w(0) = 0$.

Additionally, for the purposes of this paper we will adopt the further restriction that the Taylor-Fourier coefficients c_k of $w(z)$ around the origin satisfy the property that there is an integer $p > 0$ such that

$$\lim_{k\to\infty}\frac{|c_k|}{k^p}=0.$$

Taylor-Fourier Coefficients: given a zero-average integrable real function $f(\theta)$, one obtains from it the two sequences of Fourier coefficients, α_k from its even part and β_k from its odd part, in either case for $k \geq 1$. The Taylor-Fourier coefficients are then defined as

$$c_k = \alpha_k - \imath\beta_k,$$

so that we have the inner analytic function within the open unit disk,

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

from the real part of which $f(\theta)$ is recovered in the $\rho \to 1$ limit to the unit circle.

Fourier Conjugate: given that the inner analytic function $w(z)$ may be written within the open unit disk as

$$w(z) = f(\rho,\theta) + \imath\bar{f}(\rho,\theta),$$

where $f(\rho,\theta)$ is harmonic and $\bar{f}(\rho,\theta)$ is the harmonic conjugate function to $f(\rho,\theta)$, and that the corresponding real function $f(\theta)$ is given by

$$f(\theta) = \lim_{\rho\to 1} f(\rho,\theta),$$

we define the Fourier Conjugate real function $\bar{f}(\theta)$ to the real function $f(\theta)$ as

$$\bar{f}(\theta) = \lim_{\rho\to 1} \bar{f}(\rho,\theta).$$

The largest possible generalization of our structure is obtained if we enlarge the set of inner analytic functions to all complex analytic function within the open unit disk that have the property that $w(0) = 0$. The corresponding set of complex power series includes all such series that converge everywhere on the open unit disk and that have no constant term. There is then a corresponding generalization of the real objects on the unit circle. The new set includes not only all integrable real functions, but also singular objects we will call generalized functions, and some non-integrable real functions as well. As stated in the introduction, as a practical measure of simplification we will limit our set of inner analytic functions, for the purposes of this paper, to those whose Taylor-Fourier coefficients c_k satisfy the condition that there is an integer $p > 0$ such that

$$\lim_{k\to\infty}\frac{|c_k|}{k^p}=0. \tag{4}$$

In the next two sections we will push this generalization forward, enlarging the whole structure to include a set of singular generalized functions, as well as a class of singular non-integrable functions. As we will see, all the operations discussed here in the context of the integrable real functions can be applied to these generalized cases as well. This is due to the fact that both the normal functions and the generalized functions can be represented by inner analytic functions, so that all operations performed on either normal or generalized functions can be interpreted in terms of corresponding operations on the inner analytic functions.

3 A Set of Singular Generalized Functions

Let us start by reviewing the analysis of the representation within the open unit disk of the Dirac delta "function" that was presented in detail in [1]. Consider the very simple analytic function of z

$$w_\delta(z, z_1) = \frac{1}{2\pi} - \frac{1}{\pi} \frac{z}{z - z_1}, \tag{5}$$

where $z = \rho \exp(\imath\theta)$ and $z_1 = \exp(\imath\theta_1)$, which corresponds to the sequence of Taylor-Fourier coefficients $\alpha_k = 1/\pi$ for all $k \geq 1$, when expanded in powers of z/z_1, as one can see by the straightforward calculation of its Taylor coefficients, or by simply expanding the ratio shown in the form of a geometric series,

$$w_\delta(z, z_1) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{z}{z_1} \right)^k. \tag{6}$$

Note that the coefficients α_k correspond to a Fourier series which diverges almost everywhere. The analytic function of z shown in Equations (5) and (6) is an extended inner analytic function, that is, a rotated inner analytic function with the constant shown added, and its real part represents the Dirac delta "function" $\delta(\theta - \theta_1)$ when one takes the limit to the unit circle,

$$\begin{aligned} \delta(\theta - \theta_1) &= \lim_{\rho \to 1} \Re[w_\delta(z, z_1)] \\ &= \frac{1}{2\pi} - \frac{1}{\pi} \lim_{\rho \to 1} \frac{\rho^2 - \rho\cos(\Delta\theta)}{(\rho^2 + 1) - 2\rho\cos(\Delta\theta)}, \end{aligned}$$

where one can see that the dependence is indeed only on the difference $\Delta\theta = \theta - \theta_1$ and that the "function" is even on $\Delta\theta$. The limit represents the delta "function" in the sense that it has the following properties, as demonstrated in [1]:

1. The definition of $\delta(\theta - \theta_1)$ tends to zero when one takes the $\rho \to 1$ limit while keeping $\theta \neq \theta_1$;

2. The definition of $\delta(\theta - \theta_1)$ diverges to positive infinity when one takes the $\rho \to 1$ limit with $\theta = \theta_1$;

3. In the $\rho \to 1$ limit the integral

$$\int_a^b d\theta \, \delta(\theta - \theta_1) = 1,$$

has the value shown, for any interval $[a, b]$ which contains the point θ_1;

4. Given any continuous function $g(\theta)$, in the $\rho \to 1$ limit the integral

$$\int_a^b d\theta\, g(\theta)\delta(\theta - \theta_1) = g(\theta_1),$$

has the value shown, for any interval $[a, b]$ which contains the point θ_1.

In fact, as shown in [1], in this last property the hypothesis that $g(\theta)$ is continuous on θ can be relaxed, since $g(\theta)$, being a real function of θ, is obtained as the $\rho \to 1$ limit of a harmonic function $g(\rho, \theta)$, and it is sufficient that this last function be continuous in ρ as we make $\rho \to 1$. Also note that, although it is customary to list both separately, the third property is in fact just a particular case of the fourth property. The latter is the most important one, since it establishes the delta "function" as the integration kernel of a real-valued linear functional acting in the space of integrable real functions, defined by the integral of the product shown. This is the type of interpretation of this singular object that is established in the Schwartz theory of distributions [4].

3.1 Even and Odd Parts of the Delta "Function"

Let us take the opportunity to use the extended inner analytic function in Equation (5), as well as the structure developed in Section 2, to exemplify the determination of the extended inner analytic functions associated to the even and odd parts of the delta "function". This is, in essence, the definition of two new extended inner analytic functions and of the corresponding real singular generalized functions. In order to do this we must first determine the extended inner analytic function associated to the reflected delta "function", which is given by

$$\delta(-\theta - \theta_1) = \delta(\theta + \theta_1),$$

since the delta "function" centered at zero is even. We are now in a position to use the definitions in Equation (1) for the even and odd parts of the delta "function" $\delta(\theta - \theta_1)$. Starting from the definition of the delta function with the usual argument,

$$\begin{aligned} \delta(\theta - \theta_1) &= \lim_{\rho \to 1} \Re[w_\delta(z, z_1)], \\ w_\delta(z, z_1) &= \frac{1}{2\pi} - \frac{1}{\pi} \frac{z}{z - z_1}, \end{aligned}$$

we may now write the definition of the same function with the sign of θ_1 reversed. It is easy to see that simply exchanging the sign of θ_1 has the effect that $z_1 = \exp(\imath\theta_1)$ gets mapped onto its complex conjugate, so that we have

$$\begin{aligned} \delta(\theta + \theta_1) &= \lim_{\rho \to 1} \Re[w_\delta(z, z_1^*)], \\ w_\delta(z, z_1^*) &= \frac{1}{2\pi} - \frac{1}{\pi} \frac{z}{z - z_1^*}. \end{aligned}$$

Observe that the complex conjugation applies only to the constant z_1, so that $w_\delta(z, z_1^*)$ is still an analytic function of z. From the definitions in Equation (1) and the fact that we may simply add the corresponding inner analytic functions within the open unit disk, it follows that the extended inner analytic function $w_1(z)$ corresponding to the even part and the inner analytic function $w_2(z)$ corresponding to the odd part are given by

$$
\begin{aligned}
w_1(z) &= \frac{1}{2\pi} - \frac{z}{2\pi}\left(\frac{1}{z-z_1} + \frac{1}{z-z_1^*}\right) \\
&= \frac{1}{2\pi} - \frac{z}{2\pi}\,\frac{2z-(z_1+z_1^*)}{(z-z_1)\,(z-z_1^*)}, \\
w_2(z) &= -\frac{z}{2\pi}\left(\frac{1}{z-z_1} - \frac{1}{z-z_1^*}\right) \\
&= -\frac{z}{2\pi}\,\frac{(z_1-z_1^*)}{(z-z_1)\,(z-z_1^*)}.
\end{aligned}
$$

Note that these are not rotated inner analytic functions, but old-style ones except for the constant term of $w_1(z)$ and the lack of a factor of $\imath$ in $w_2(z)$, so that we have that $w_1(0) = 1/(2\pi)$ and $w_2(0) = 0$, and also that $w_1(z)$ reduces to a real function on the interval $(-1, 1)$ of the real axis, while $w_2(z)$ reduces to a purely imaginary function on that same interval. Therefore, taking the limit to the unit circle, we may conclude that the even and odd parts of the delta "function" are represented by

$$
\begin{aligned}
\mathfrak{E}[\delta(\theta-\theta_1)] &= \frac{\delta(\theta-\theta_1)+\delta(\theta+\theta_1)}{2} \\
&= \frac{1}{2\pi} - \frac{1}{2\pi}\lim_{\rho\to1}\Re\left[z\,\frac{2z-(z_1+z_1^*)}{(z-z_1)\,(z-z_1^*)}\right], \\
\mathfrak{O}[\delta(\theta-\theta_1)] &= \frac{\delta(\theta-\theta_1)-\delta(\theta+\theta_1)}{2} \\
&= -\frac{1}{2\pi}\lim_{\rho\to1}\Re\left[z\,\frac{(z_1-z_1^*)}{(z-z_1)\,(z-z_1^*)}\right].
\end{aligned}
$$

Due to the analyticity of the functions involved, these limits exist everywhere except at the two points of singularity z_1 and z_1^*. This illustrates how one can use the inner analytic functions to define generalized functions on the unit circle, in a very simple way, making use of the vector-space properties of the inner analytic functions, when defined according to our new standard convention.

3.2 Derivatives of the Delta "Function"

Starting once gain from the extended inner analytic function associated to the delta "function" $\delta(\theta-\theta_1)$, shown in Equation (5), we may now define from it further generalized functions with the use of the operation of integration by parts. In order to do this we write $w_\delta(z, z_1)$ as

$$
w_\delta(z, z_1) = \frac{1}{2\pi} + f(\rho,\theta,\theta_1) + \imath\bar{f}(\rho,\theta,\theta_1),
$$

and consider integrals on circles of radius $\rho < 1$ centered at the origin involving a function $g(\rho, \theta)$. Since this will always be the real part of the inner analytic function corresponding to some real function $g(\theta)$, it follows that within the open unit disk it is infinitely differentiable on both arguments. Observe that the following integrals are *not* integrals of an analytic function on a closed contour, but rather a pair of real integrals of real functions on the circles, one integral in the real part and one in the imaginary part,

$$
\begin{aligned}
I_\delta(\rho,\theta_1) &= \int_{-\pi}^{\pi} d\theta\,\frac{dg(\rho,\theta)}{d\theta}\,w_\delta(z,z_1) \\
&= \int_{-\pi}^{\pi} d\theta\,\frac{dg(\rho,\theta)}{d\theta}\left[\frac{1}{2\pi} + f(\rho,\theta,\theta_1) + \imath\bar{f}(\rho,\theta,\theta_1)\right]. \qquad (7)
\end{aligned}
$$

Since on the circles we have $dz = \imath z d\theta$, these integrals within the open unit disk can also be written as

$$I_\delta(\rho,\theta_1) = \oint_C dz\,\frac{dg(\rho,\theta)}{dz}\,w_\delta(z,z_1),$$

where C is any circle with radius $\rho < 1$ centered at the origin and where $g(\rho,\theta)$ is a *real* function of ρ and θ. In either case, since the circle has no boundary, it is clear that we may integrate by parts without generating an integrated term, and thus obtain

$$\begin{aligned} I_\delta(\rho,\theta_1) &= -\oint_C dz\,g(\rho,\theta)\,\frac{d}{dz}w_\delta(z,z_1) \\ &= -\int_{-\pi}^{\pi} d\theta\,g(\rho,\theta)\left[\frac{d}{d\theta}f(\rho,\theta,\theta_1) + \imath\,\frac{d}{d\theta}\bar{f}(\rho,\theta,\theta_1)\right]. \end{aligned} \tag{8}$$

Note that the integration by parts can be written in either one of these two equivalent ways, within the open unit disk,

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta\,\frac{dg(\rho,\theta)}{d\theta}\,w_\delta(\rho,\theta,\theta_1) &= -\int_{-\pi}^{\pi} d\theta\,g(\rho,\theta)\,\frac{dw_\delta(\rho,\theta,\theta_1)}{d\theta}, \\ \oint_C dz\,\frac{dg(\rho,\theta)}{dz}\,w_\delta(z,z_1) &= -\oint_C dz\,g(\rho,\theta)\,\frac{dw_\delta(z,z_1)}{dz}, \end{aligned}$$

where C is any circle with radius $\rho < 1$ centered at the origin and where $w_\delta(z,z_1)$ can be written either in terms of z or in terms of ρ and θ. We now observe that the real part of the original form of this integral in Equation (7), when the limit $\rho \to 1$ to the unit circle is taken, becomes the integral of the product of the delta "function" and the derivative of the real function, so that we have

$$\begin{aligned} I_\delta(1,\theta_1) &= \int_{-\pi}^{\pi} d\theta\left[\frac{dg(1,\theta)}{d\theta}\,\delta(\theta-\theta_1) + \imath\,\frac{dg(1,\theta)}{d\theta}\,\bar{f}(1,\theta,\theta_1)\right] \\ &= \frac{dg}{d\theta}(1,\theta_1) + \imath\int_{-\pi}^{\pi} d\theta\,\frac{dg(1,\theta)}{d\theta}\,\bar{f}(1,\theta,\theta_1), \end{aligned}$$

where we used the integration properties of $\delta(\theta-\theta_1)$. If we also take the limit to the unit circle of the last expression in Equation (8), then it follows that on that circle we have

$$\begin{aligned} I_\delta(1,\theta_1) &= -\lim_{\rho\to 1}\int_{-\pi}^{\pi} d\theta\,g(\rho,\theta)\left[\frac{d}{d\theta}f(\rho,\theta,\theta_1) + \imath\,\frac{d}{d\theta}\bar{f}(\rho,\theta,\theta_1)\right] \\ &= -\int_{-\pi}^{\pi} d\theta\,g(1,\theta)\lim_{\rho\to 1}\left[\frac{d}{d\theta}f(\rho,\theta,\theta_1)\right] - \imath\int_{-\pi}^{\pi} d\theta\,g(1,\theta)\,\lim_{\rho\to 1}\left[\frac{d}{d\theta}\bar{f}(\rho,\theta,\theta_1)\right]. \end{aligned}$$

From these two expressions for $I_\delta(1,\theta_1)$ we conclude that we have

$$\begin{aligned} &\frac{dg}{d\theta}(1,\theta_1) + \imath\int_{-\pi}^{\pi} d\theta\,\frac{dg(1,\theta)}{d\theta}\,\bar{f}(1,\theta,\theta_1) \\ &= -\int_{-\pi}^{\pi} d\theta\,g(1,\theta)\lim_{\rho\to 1}\left[\frac{d}{d\theta}f(\rho,\theta,\theta_1)\right] - \imath\int_{-\pi}^{\pi} d\theta\,g(1,\theta)\lim_{\rho\to 1}\left[\frac{d}{d\theta}\bar{f}(\rho,\theta,\theta_1)\right]. \end{aligned}$$

The real part of this equality tell us that the expression involving the limit of the derivative shown within the integral at the right-hand side can be interpreted as a definition of the first derivative of the delta "function",

$$\frac{dg}{d\theta}(1,\theta_1) = -\int_{-\pi}^{\pi} d\theta\, g(1,\theta) \lim_{\rho\to 1}\left[\frac{d}{d\theta} f(\rho,\theta,\theta_1)\right] \quad \Rightarrow$$

$$\frac{d}{d\theta}\delta(\theta-\theta_1) \equiv \lim_{\rho\to 1}\left[\frac{d}{d\theta} f(\rho,\theta,\theta_1)\right].$$

This is a new generalized function, that has the property that through integration over the unit circle it attributes to a given differentiable function minus the value of its derivative at the point θ_1,

$$\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\, \frac{d}{d\theta}\delta(\theta-\theta_1) = -\frac{dg}{d\theta}(1,\theta_1).$$

It is, therefore, the integration kernel of a real-valued linear functional acting in the space of integrable real functions. As we will see shortly, this new generalized function is indeed dependent only on the difference $\theta - \theta_1$. We will adopt for this new singular object the notation

$$\delta'(\theta-\theta_1) \equiv \frac{d}{d\theta}\delta(\theta-\theta_1).$$

We may now determine the inner analytic function associated to this new singular object. Since derivatives with respect to θ may be written as logarithmic derivatives of the corresponding inner analytic function within the open unit disk, as shown in [2], we have, using the notation established in that paper,

$$\begin{aligned} w_{\delta'}(z,z_1) &= w_{\delta}^{\bullet}(z,z_1) \\ &= \imath z \frac{d}{dz} w_{\delta}(z,z_1) \\ &= \frac{d}{d\theta} f(\rho,\theta,\theta_1) + \imath \frac{d}{d\theta}\bar{f}(\rho,\theta,\theta_1), \end{aligned}$$

so that the generalized function $\delta'(\theta-\theta_1)$ is given by the limit to the unit circle of the real part of the inner analytic function $w_{\delta'}(z,z_1)$,

$$\delta'(\theta-\theta_1) = \lim_{\rho\to 1} \Re[w_{\delta'}(z,z_1)].$$

Since we have the inner analytic function that corresponds to the delta "function" in explicit form, we may perform an explicit calculation in order to obtain the inner analytic function that corresponds to the derivative of the delta "function",

$$\begin{aligned} \imath z \frac{d}{dz} w_{\delta}(z,z_1) &= \imath z \frac{d}{dz}\left[\frac{1}{2\pi} - \frac{1}{\pi}\frac{z}{z-z_1}\right] \\ &= \frac{-\imath z}{\pi}\frac{d}{dz}\left[\frac{z}{z-z_1}\right] \\ &= \frac{-\imath z}{\pi}\left[\frac{1}{z-z_1} - \frac{z}{(z-z_1)^2}\right] \\ &= \frac{-\imath z}{\pi}\left[\frac{z-z_1}{(z-z_1)^2} - \frac{z}{(z-z_1)^2}\right] \quad \Rightarrow \end{aligned}$$

$$w_{\delta'}(z,z_1) = \frac{\imath z z_1}{\pi(z-z_1)^2}.$$

It is not difficult to verify that this inner analytic function corresponds to the sequence of Taylor-Fourier coefficients $\alpha_k = k/\pi$ for all $k \geq 1$, when expanded in powers of z/z_1, by simply taking the logarithmic derivative of Equation (6). Therefore, the corresponding Fourier series also diverges almost everywhere, and does so faster than the Fourier series of the delta "function". This derivation is an example of the derivative of a singular and therefore in principle non-differentiable object on the unit circle being regularized, and in fact defined, within the open unit disk of the complex plane. Following the pattern of the inner analytic function associated to the delta "function", which has a simple pole on the unit circle, this one has a second-order pole at the same point. Rationalizing this function, in order to identify its real and imaginary parts, we get, with $z = \rho\exp(\imath\theta)$, $z_1 = \exp(\imath\theta_1)$ and $\Delta\theta = \theta - \theta_1$,

$$\begin{aligned}
\imath z\,\frac{d}{dz} w_\delta(z, z_1) &= \frac{\imath\rho\left(\rho^2\,\mathrm{e}^{-\imath\Delta\theta} - 2\rho + \mathrm{e}^{\imath\Delta\theta}\right)}{\pi\left[(\rho^2+1) - 2\rho\cos(\theta-\theta_1)\right]^2} \\
&= \frac{\rho\left(\rho^2-1\right)\sin(\theta-\theta_1)}{\pi\left[(\rho^2+1) - 2\rho\cos(\theta-\theta_1)\right]^2} + \\
&\quad +\imath\,\frac{\rho\left[-2\rho + \left(\rho^2+1\right)\cos(\theta-\theta_1)\right]}{\pi\left[(\rho^2+1) - 2\rho\cos(\theta-\theta_1)\right]^2} \quad \Rightarrow \\
\delta'(\theta-\theta_1) &= \lim_{\rho\to 1} \frac{\rho\left(\rho^2-1\right)\sin(\theta-\theta_1)}{\pi\left[(\rho^2+1) - 2\rho\cos(\theta-\theta_1)\right]^2}.
\end{aligned}$$

This gives us an explicit representation of the generalized function $\delta'(\theta-\theta_1)$, which is thus seen to depend only on $\Delta\theta = \theta-\theta_1$, as a limit to the unit circle. Observe that the expression on the right-hand side changes sign with $\Delta\theta$, and is therefore odd with respect to $\Delta\theta$.

The process of integration by parts used to define this generalized function can now be iterated, since we may consider the integral

$$\begin{aligned}
I_{\delta'}(\rho, \theta_1) &= \oint_C dz\,\frac{dg(\rho,\theta)}{dz}\,w_{\delta'}(z, z_1) \\
&= \int_{-\pi}^{\pi} d\theta\,\frac{dg(\rho,\theta)}{d\theta}\,w_{\delta'}(z, z_1) \\
&= \int_{-\pi}^{\pi} d\theta\,\frac{dg(\rho,\theta)}{d\theta}\left[f'(\rho,\theta,\theta_1) + \imath\bar{f}'(\rho,\theta,\theta_1)\right], \qquad (9)
\end{aligned}$$

which once more can be integrated by parts to yield

$$\begin{aligned}
I_{\delta'}(\rho, \theta_1) &= -\oint_C dz\,g(\rho,\theta)\,\frac{d}{dz} w_{\delta'}(z, z_1) \\
&= -\int_{-\pi}^{\pi} d\theta\,g(\rho,\theta)\left[\frac{d}{d\theta} f'(\rho,\theta,\theta_1) + \imath\,\frac{d}{d\theta}\bar{f}'(\rho,\theta,\theta_1)\right]. \qquad (10)
\end{aligned}$$

We now observe that the real part of the original form of this integral in Equation (9), when the limit to the unit circle is taken, becomes the integral of the product of the first derivative of the delta "function" and the derivative of the real function, so that we have

$$\begin{aligned}
I_{\delta'}(1, \theta_1) &= \int_{-\pi}^{\pi} d\theta\left[\frac{dg(1,\theta)}{d\theta}\,\delta'(\theta-\theta_1) + \imath\,\frac{dg(1,\theta)}{d\theta}\,\bar{f}'(1,\theta,\theta_1)\right] \\
&= -\frac{d^2 g}{d\theta^2}(1,\theta_1) + \imath\int_{-\pi}^{\pi} d\theta\,\frac{dg(1,\theta)}{d\theta}\,\bar{f}'(1,\theta,\theta_1),
\end{aligned}$$

where we used the integration properties of $\delta'(\theta-\theta_1)$. If we also take the limit to the unit circle of the last expression in Equation (10), then it follows that on that circle we have

$$\begin{aligned} I_{\delta'}(1,\theta_1) &= -\lim_{\rho\to 1}\int_{-\pi}^{\pi} d\theta\, g(\rho,\theta) \left[\frac{d}{d\theta}f'(\rho,\theta,\theta_1)+\imath\,\frac{d}{d\theta}\bar{f}'(\rho,\theta,\theta_1)\right] \\ &= -\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\lim_{\rho\to 1}\left[\frac{d}{d\theta}f'(\rho,\theta,\theta_1)\right]-\imath\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\,\lim_{\rho\to 1}\left[\frac{d}{d\theta}\bar{f}'(\rho,\theta,\theta_1)\right]. \end{aligned}$$

From these two expressions for $I_{\delta'}(1,\theta_1)$ we conclude that we have

$$\begin{aligned} &-\frac{d^2g}{d\theta^2}(1,\theta_1)+\imath\int_{-\pi}^{\pi} d\theta\,\frac{dg(1,\theta)}{d\theta}\,\bar{f}'(1,\theta,\theta_1) \\ &= -\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\lim_{\rho\to 1}\left[\frac{d}{d\theta}f'(\rho,\theta,\theta_1)\right]-\imath\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\,\lim_{\rho\to 1}\left[\frac{d}{d\theta}\bar{f}'(\rho,\theta,\theta_1)\right]. \end{aligned}$$

This time the real part of this equality tell us that the expression involving the limit of the derivative shown within the integral at the right-hand side can be interpreted as a definition of the second derivative of the delta "function",

$$\frac{d^2g}{d\theta^2}(1,\theta_1)=\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\lim_{\rho\to 1}\left[\frac{d}{d\theta}f'(\rho,\theta,\theta_1)\right] \quad\Rightarrow$$

$$\frac{d^2}{d\theta^2}\delta(\theta-\theta_1)\equiv\lim_{\rho\to 1}\left[\frac{d}{d\theta}f'(\rho,\theta,\theta_1)\right].$$

This is a new generalized function, that has the property that through integration over the unit circle it attributes to a given twice-differentiable function the value of its second derivative at the point θ_1,

$$\int_{-\pi}^{\pi} d\theta\, g(1,\theta)\,\frac{d^2}{d\theta^2}\delta(\theta-\theta_1)=\frac{d^2g}{d\theta^2}(1,\theta_1).$$

Once again we see that this is the integration kernel of a real-valued linear functional acting in the space of integrable real functions. We will adopt for this new singular object the notation

$$\delta''(\theta-\theta_1)\equiv\frac{d^2}{d\theta^2}\delta(\theta-\theta_1).$$

We may now determine the inner analytic function associated to this new singular object. Once again, since derivatives with respect to θ may be written as logarithmic derivatives of the corresponding inner analytic function within the open unit disk, as shown in [2], we have, using the notation established in that paper,

$$\begin{aligned} w_{\delta''}(z,z_1) &= w^{\bullet}_{\delta'}(z,z_1) \\ &= \imath z\,\frac{d}{dz}w_{\delta'}(z,z_1) \\ &= \frac{d}{d\theta}f'(\rho,\theta,\theta_1)+\imath\frac{d}{d\theta}\bar{f}'(\rho,\theta,\theta_1), \end{aligned}$$

so that the generalized function $\delta''(\theta-\theta_1)$ is given by the limit to the unit circle of the inner analytic function $w_{\delta''}(z,z_1)$,

$$\delta''(\theta-\theta_1)=\lim_{\rho\to 1}\Re[w_{\delta''}(z,z_1)].$$

Since we have the inner analytic function that corresponds to the first derivative of the delta "function" in explicit form, we have the capability to perform an explicit calculation in order to get the inner analytic function that corresponds to the second derivative of the delta "function". It can be easily verified that the inner analytic function $w_{\delta''}(z,z_1)$ corresponds to the sequence of Taylor-Fourier coefficients $\alpha_k = k^2/\pi$ for all $k \geq 1$, when expanded in powers of z/z_1.

Just as the first and the second logarithmic derivatives of the inner analytic function corresponding to the original delta "function" represent the first and second derivatives of the delta "function" within the open unit disk, so it is with all the higher derivatives. By iterating repeatedly this process of integration by parts, we can generate a whole infinite sequence of singular generalized functions and their corresponding inner analytic functions. Iterating one more time one gets for the first few elements of this sequence of inner analytic functions the list of results that follows,

$$\begin{aligned} w_{\delta^{1\prime}}(z,z_1) &= -\frac{1}{\pi\boldsymbol{\imath}^1}\,\frac{zz_1}{(z-z_1)^2}, \\ w_{\delta^{2\prime}}(z,z_1) &= -\frac{1}{\pi\boldsymbol{\imath}^2}\,\frac{z(z+z_1)z_1}{(z-z_1)^3}, \\ w_{\delta^{3\prime}}(z,z_1) &= -\frac{1}{\pi\boldsymbol{\imath}^3}\,\frac{z\left(z^2+4zz_1+z_1^2\right)z_1}{(z-z_1)^4}. \end{aligned}$$

Generalizing to the $n^{\rm th}$ derivative, we see that the manifestly systematic factors in $w_{\delta^{n\prime}}(z,z_1)$ can be written as

$$-\frac{zz_1}{\pi\boldsymbol{\imath}^n(z-z_1)^{n+1}},$$

and the remaining factors are polynomials on z. Note that the logarithmic derivation operator conserves the homogeneity of the powers of z and z_1, because we have that

$$\boldsymbol{\imath} z\,\frac{d}{dz}\,\_ = \boldsymbol{\imath}(z/z_1)\,\frac{d}{d(z/z_1)}\,\_\,.$$

Therefore all these inner analytic functions can be written as functions of only z/z_1, so that they are all rotated inner analytic functions, with the angle θ_1. Also, it is not difficult to verify that $w_{\delta^{n\prime}}(z,z_1)$ corresponds to the sequence of Taylor-Fourier coefficients $\alpha_k = k^n/\pi$ for all $k \geq 1$, when expanded in powers of z/z_1. It is possible to work out a general recursion formula for the inner analytic functions corresponding to all the multiple derivatives of the delta "function". In order to do this we may write, in terms of the variable $\chi = z/z_1$, for $n > 0$, that

$$w_{\delta^{n\prime}}(\chi) = -\frac{1}{\pi\boldsymbol{\imath}^n}\,\frac{\chi P_{n-1}(\chi)}{(\chi-1)^{n+1}},$$

where $P_{n-1}(\chi)$ is a polynomial of order $n-1$ on χ, with real (in fact, integer) coefficients $A_{n,i}$,

$$P_n(\chi) = \sum_{i=0}^{n} A_{n,i}\chi^i.$$

One can see that the first few polynomials, for $n > 0$, are therefore given by

$i \Longrightarrow$	0	1	2	3	4	5	$\cdots$
$n=0$	1	0	0	0	0	0	$\cdots$
$n=1$	1	1	0	0	0	0	$\cdots$
$n=2$	1	4	1	0	0	0	$\cdots$
$n=3$	1	11	11	1	0	0	$\cdots$
$n=4$	1	26	66	26	1	0	$\cdots$
$n=5$	1	57	302	302	57	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1: A table of values for the coefficients $A_{n,i}$.

$$\begin{aligned} P_0(\chi) &= 1, \\ P_1(\chi) &= 1+\chi, \\ P_2(\chi) &= 1+4\chi+\chi^2. \end{aligned}$$

As is shown in Appendix B, one obtains for the coefficients $A_{n,i}$ the purely algebraic two-index recurrence relation

$$A_{n,i} = (i+1)A_{n-1,i} + (n-i+1)A_{n-1,i-1}, \tag{11}$$

for $0 < i < n$. When using this formula we should assume that $A_{n,i} = 0$ for $i < 0$ and for $i > n$, and we must also use the initial value $A_{0,0} = 1$. Using this recursion relation we may construct the table of values of $A_{n,i}$ shown in Table 1. Using this method we are therefore able to determine completely the inner analytic function associated to the derivative of any given order of the delta "function".

Note that the $n^{\rm th}$ derivative of the delta "function" is represented by an inner analytic function with a pole of order $n+1$ on the unit circle, and that, as we discussed before, it is associated to the sequence of Taylor-Fourier coefficients $\alpha_k = k^n/\pi$ for all $k \geq 1$, when expanded in powers of χ. We may therefore conclude that all these generalized functions are within the set defined by the limited definition discussed in the introduction, and expressed by Equations (2) and (4), including therefore the requirement that the sequence of Taylor-Fourier coefficients α_k not diverge faster than all powers of k when $k \to \infty$.

4 A Class of Locally Non-Integrable Functions

In we consider once again the extended inner analytic function $w_\delta(z, z_1)$ associated to the delta "function", given in Equation (5), we realize that not only its real part is a representation of the delta "function" in the $\rho \to 1$ limit, but the corresponding FC function is also representable by the same extended inner analytic function, as the $\rho \to 1$ limit of its imaginary part. This is a very simple normal function $\bar{f}(\theta - \theta_1)$, although a singular and non-integrable one, which was given in the appendices of [1], and which we repeat here,

$$\bar{f}(\theta - \theta_1) = \frac{1}{\pi} \frac{1 + \cos(\theta - \theta_1)}{2\sin(\theta - \theta_1)}.$$

This function has a singularity that behaves as $1/(\theta - \theta_1)$ around the point θ_1, which is not, therefore, an integrable singularity. This means that the integral of this function over the unit circle is not well defined. However, it is still associated to the Taylor-Fourier coefficients of the delta "function", and can be recovered as the $\rho \to 1$ limit of the imaginary part of the same extended inner analytic function. We may also recover this function as the real part of its own extended inner analytic function, which is simply given by

$$w(z, z_1) = -\imath w_\delta(z, z_1),$$

being thus defined according to our new criterion. It follows from this fact that it is not really necessary for a real function to be integrable in order for it to have a well-defined sequence of Taylor-Fourier coefficients and to be recoverable almost everywhere from, and therefore representable almost everywhere by, an inner analytic function. In fact, it is possible to identify without too much trouble a whole class of singular non-integrable real functions that can be represented by inner analytic functions. All we have to do in order to accomplish this is to find a consistent way to associate a sequence of Fourier coefficients to such functions.

We will call functions that are locally integrable almost everywhere, by which we mean everywhere but in the neighborhoods of a finite set of singular points on the unit circle, by the name of *locally non-integrable functions.* This means that the function is integrable in all closed intervals within the unit circle that do *not* contain a singular point, of which we assume there is a finite number. We will be able to define Fourier coefficients for such functions so long as they can be obtained as derivatives of any finite order of integrable functions. Let us start the argument with the case of the first derivative. If a real function $f(\theta)$ is locally non-integrable, so that the coefficients

$$\begin{aligned} \alpha_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f(\theta), \\ \beta_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f(\theta), \end{aligned}$$

for $k \geq 1$, do not exist if defined in this way, due to the fact that the integrals do not exist, but $f(\theta)$ is the derivative with respect to θ of a zero-average real function $f^{-1\prime}(\theta)$ which is defined almost everywhere and integrable on the whole unit circle, then we may define the Taylor-Fourier coefficients associated to $f(\theta)$ as

$$\begin{aligned} \alpha_k &= \frac{k}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f^{-1\prime}(\theta), \\ \beta_k &= -\frac{k}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f^{-1\prime}(\theta), \end{aligned}$$

for $k \geq 1$, where we note that the sine and cosine have been interchanged. We might call this the *extended definition* of the Fourier coefficients. Note that if $f(\theta)$ is the derivative of a zero-average function, then it also has no constant term in its Fourier expansion, and thus we may consider it to be zero-average as well. Note also that when $f(\theta)$ is integrable and the usual definitions of α_k and β_k are in effect, we may always write that

$$\begin{aligned} \alpha_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f(\theta) \\ &= \frac{k}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f^{-1\prime}(\theta), \\ \beta_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f(\theta) \\ &= -\frac{k}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f^{-1\prime}(\theta), \end{aligned}$$

for $k \geq 1$, since we can always integrate by parts on the unit circle, without generating an integrated term. The point is that even if $f(\theta)$ is not integrable and the usual definition does not work, the extended definition may still exist. This will be the case when $f(\theta)$, though not an integrable function itself, is the derivative of an integrable function. This will often happen around singular points where $f(\theta)$ diverges to infinity. Another way to implement this extended definition is to write the Fourier coefficients α_k and β_k of the function $f(\theta)$ in terms of the Fourier coefficients $\alpha_k^{-1\prime}$ and $\beta_k^{-1\prime}$ of its zero-average first primitive $f^{-1\prime}(\theta)$,

$$\begin{aligned} \alpha_k &= k\beta_k^{-1\prime}, \\ \beta_k &= -k\alpha_k^{-1\prime}. \end{aligned}$$

The best way to interpret this extended definition is in terms of the corresponding inner analytic functions within the open unit disk. In order to do this, one starts with a locally non-integrable function $f(\theta)$. Since it is integrable in all closed intervals within the unit circle that do *not* contain a singularity, one may in principle integrate within these intervals in order to find a zero-average primitive $f^{-1\prime}(\theta)$. It should be possible, in this way, to determine a zero-average real function $f^{-1\prime}(\theta)$ such that $f(\theta)$ is its derivative almost everywhere. So long as $f^{-1\prime}(\theta)$ is integrable, one may then calculate its Taylor-Fourier coefficients in the usual way, and thus construct the corresponding inner analytic function $w^{-1\cdot}(z)$. By taking the logarithmic derivative of this function one then gets the inner analytic function $w(z)$ that corresponds to $f(\theta)$, and at the same time determines its Taylor-Fourier coefficients.

Note that if $f(\theta)$ is not integrable then the Fourier series associated to its extended Fourier coefficients will typically diverge almost everywhere, due to the extra factor of k in the coefficients. However, so long as the coefficients do not diverge faster than all powers of k, the corresponding inner analytic function can still be constructed with those coefficients, and the function can still be recovered from it almost everywhere. Therefore, it is still true that the locally non-integrable real function is uniquely characterized by its extended Fourier coefficients almost everywhere.

The definition of the Fourier coefficients can be further extended to cases in which both $f(\theta)$ and $f^{-1\prime}(\theta)$ are locally non-integrable, so long as $f(\theta)$ is the second derivative with respect to θ of a zero-average integrable real function $f^{-2\prime}(\theta)$. In this case we may define

$$\begin{aligned} \alpha_k &= -\frac{k^2}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f^{-2\prime}(\theta), \\ \beta_k &= -\frac{k^2}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f^{-2\prime}(\theta), \end{aligned}$$

for $k \geq 1$, by simply using for $f^{-1\prime}(\theta)$ the same argument used previously for $f(\theta)$. Note that the sine and cosine have now been interchanged back to their original positions. With the use of repeated integrations by parts this can be extended to locally non-integrable functions $f(\theta)$ which are the derivatives of any finite order of integrable real functions with finite numbers of singularities where they diverge to infinity. It is not difficult to derive general formulas for the coefficients α_k and β_k valid for the case of the derivative of order n, and the corresponding partial integrations until the primitive of order n is produced. We just have to systematize the first few cases, which are given by

$$\begin{aligned} \alpha_k &= +\frac{k^0}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f^{-0\prime}(\theta) \\ &= +\frac{k^1}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f^{-1\prime}(\theta) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{k^2}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-2\prime}(\theta) \\
&= -\frac{k^3}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-3\prime}(\theta) \\
&= +\frac{k^4}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-4\prime}(\theta) \\
&= +\frac{k^5}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-5\prime}(\theta), \\
\beta_k &= +\frac{k^0}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-0\prime}(\theta) \\
&= -\frac{k^1}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-1\prime}(\theta) \\
&= -\frac{k^2}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-2\prime}(\theta) \\
&= +\frac{k^3}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-3\prime}(\theta) \\
&= +\frac{k^4}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-4\prime}(\theta) \\
&= -\frac{k^5}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-5\prime}(\theta),
\end{aligned}
$$

where we used the notation $f(\theta) = f^{-0\prime}(\theta)$. In order to systematize these cases we must separate then into those in which n is even and those in which n is odd. For even n we make $n = 2j$, with $j = 0, 1, 2, \ldots, \infty$, and we may write

$$
\begin{aligned}
\alpha_k &= \frac{(-1)^j k^n}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-n\prime}(\theta), \\
\beta_k &= \frac{(-1)^j k^n}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-n\prime}(\theta),
\end{aligned}
$$

for $k \geq 1$, while for odd n we make $n = 2j+1$, with $j = 0, 1, 2, \ldots, \infty$, and we may write

$$
\begin{aligned}
\alpha_k &= \frac{(-1)^j k^n}{\pi}\int_{-\pi}^{\pi} d\theta\,\sin(k\theta) f^{-n\prime}(\theta), \\
\beta_k &= \frac{(-1)^{j+1} k^n}{\pi}\int_{-\pi}^{\pi} d\theta\,\cos(k\theta) f^{-n\prime}(\theta), \qquad (12)
\end{aligned}
$$

for $k \geq 1$, where we note the interchange of the sine and the cosine, as well as the extra sign in the second equation. These equations constitute the most general definition of the *extended Taylor-Fourier coefficients* of the function $f(\theta)$. The idea is that we use for the calculation of the coefficients the smallest value of n for which the integrals exist, that is, the smallest value of n for which the primitive of order n of $f(\theta)$, that is the function $f^{-n\prime}(\theta)$, is a zero-average integrable function.

Since the Fourier coefficients of an integrable function are always limited, it is clear that the extended Fourier coefficients of the class of locally non-integrable functions that we are considering here, which are given by those limited coefficients multiplied by a fixed power of k, will never diverge to infinity faster than all powers of k as we make $k \to \infty$. This is in accordance of our restrictive hypothesis, given in Equation (4), and hence we see that with the use of the extended coefficients we are always able to represent this class of locally non-integrable functions by means of inner analytic functions within the open unit disk. We are, therefore, always able to recover these functions almost everywhere as limits to the unit circle of the real part of the corresponding inner analytic functions, although the corresponding Fourier series are always divergent almost everywhere.

As a final note, let us observe that, although the extended Fourier coefficients of non-integrable functions will always result in divergent Fourier series, using the analytic structure within the open unit disk one may apply to these coefficients the technique of *singularity factorization* developed in [2], in order to obtain convergent trigonometric series to represent these functions everywhere on the unit circle but at the singular points. In fact, several such series can be constructed from the extended Fourier coefficients, with increasing levels of speed of convergence. These are the *center series* which were developed and described in [2], and tested numerically in [6]. In this way, the extended Fourier coefficients are seen to still have an algorithmic value, in terms of the numerical representation of locally non-integrable functions, despite the fact that the corresponding Fourier series are divergent.

5 Boundary Value Problems on the Unit Disk

Let us show that the correspondence between inner analytic functions within the open unit disk and real functions on the unit circle can be interpreted in terms of boundary value problems of the two-dimensional Laplace equation on the unit disk. In order to do this, let us describe how one goes about finding the inner analytic function that corresponds to a given integrable real function. We will formulate the ideas in terms of normal integrable real functions, and later generalize the results that are found. The idea is to develop a criterion for the existence of the inner analytic function associated to a normal integrable real function, that can later be applied to both normal and generalized function, as well as to both regular and singular functions.

Let an arbitrary integrable real function $f(\theta)$ be given. According to the results we obtained before, the corresponding inner analytic function can be constructed in the following way: since this real function is integrable on the unit circle, one can calculate the corresponding Taylor-Fourier coefficients and then use them to build the corresponding inner analytic function by means of a convergent complex Taylor series. This produces an inner analytic function that can be written as

$$w(z) = f(\rho, \theta) + \imath \bar{f}(\rho, \theta),$$

where $f(\rho,\theta)$ is harmonic and $\bar{f}(\rho,\theta)$ is its harmonic conjugate. The process described above can be understood as a way to determine $f(\rho,\theta)$ from $f(\theta)$. If $f(\rho,\theta)$ can be defined as a harmonic function on the open unit disk, then its harmonic conjugate $\bar{f}(\rho,\theta)$ necessarily exists and is harmonic on the same domain. This is so because a harmonic function defined on a simply connected domain within the complex plane always admits a harmonic conjugate. In fact, in our case here the harmonic conjugate is given by a real line integral within the open unit disk, written here both in Cartesian coordinates and in a somewhat more abstract way using vector notation,

$$\begin{aligned}
\bar{f}(x,y) &= \int_{(0,0)}^{(x,y)} \left[\frac{\partial f(x',y')}{\partial x'}\, dy' - \frac{\partial f(x',y')}{\partial y'}\, dx' \right] \\
&= \int_{0}^{z} \left[\vec{\nabla} f(z') \times d\vec{z}' \right]_n ,
\end{aligned}$$

where what we have here is actually the component of a vector product normal to the complex plane, and where the infinitesimal vector is defined as $\vec{dz} = (dx, dy, 0)$. Therefore, we may focus our attention on the determination of $f(\rho,\theta)$, aiming at describing it in a more general way, so as not to depend on the determination of the Fourier coefficients by means of integrals over the unit circle. Now, the problem of finding $f(\rho,\theta)$ within the open unit disk starting from the values of $f(\theta)$ on the unit circle can be understood as a standard *boundary value problem* on the unit disk. In fact, since the real and imaginary parts of an analytic function are harmonic functions, it is the problem of finding a solution of the two-dimensional Laplace equation, written here in polar coordinates,

$$\frac{\partial^2 f(\rho,\theta)}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 f(\rho,\theta)}{\partial\theta^2} = 0,$$

valid within the open unit disk, that satisfies the boundary condition, over the whole unit circle,

$$f(1,\theta) = f(\theta).$$

This is an instance of the famous Dirichlet problem, to wit the Dirichlet problem on the unit disk [7]. The simplest theorem of existence and uniqueness of the solution of boundary value problems such as this one states that the solution exists and is unique so long as the boundary condition $f(\theta)$ is well defined and continuous over the whole boundary. There are further theorems that establish the existence and uniqueness of the solution for boundary conditions $f(\theta)$ that are well defined everywhere but discontinuous at some points. There seems to be no well-established general theorems about the solution of the boundary value problem when the real function $f(\theta)$ has points of singularity where it diverges to infinity.

The existence or non-existence of the solution of the boundary value problem is a very general way to determine whether or not the inner analytic function can be defined. We may consider applying it not only to integrable normal real functions, but also to locally non-integrable normal real functions, as well as to singular generalized functions. In these two singular cases the main difference is the existence of points of singularity where the quantity $f(\theta)$ diverges to infinity. In these cases it is not enough to apply as boundary conditions the known values of the functions at the unit circle, since this clearly cannot be done at the singular points. The set of boundary conditions are thus seen to be incomplete in such cases. If we are to have a complete set of boundary conditions, the boundary condition at every point of divergence of $f(\theta)$ must be replaced by an individual extra condition, for example a condition such as the one about the integral of the delta "function".

This leads one to think of a modified boundary value problem, in which the continuously infinite set of conditions on the boundary over the whole unit circle is replaced by a boundary condition with a certain number of point-like "holes". We might call this new type of boundary conditions "punctured boundary conditions". In such cases one must add to the boundary-value problem a set of extra conditions, in the same number as the holes. In the case of normal real functions that diverge to infinity at the singular points, the extra conditions might be statements about the form or speed of the divergence. This implies that the behavior of the real functions in neighborhoods around the singular points is known and can be used in this way, as local conditions around each singular point.

For more singular objects such as the delta "function" the value of the "function" in such neighborhoods is simply zero, except at the singular points themselves, so that information regarding how the "function" tends to infinity is not available on the unit circle and thus cannot be used to characterize the "functions". Therefore, one is left only with the alternative of imposing a global rather than local extra condition, such as the one involving the integral of the delta "function". Seen in this light, the Dirac delta "function", and the other singular generalized functions associated to it that we examined in Section 3, are all solutions of punctured boundary value problems of the Laplace equation on the unit disk, with a single singular point z_1 on the unit circle.

We are led therefore to the fact that there is an interesting side-effect of the correspondence between inner analytic functions within the open unit disk and real objects on the unit circle. The structure we introduced at once implies some quite general existence and uniqueness theorems for the solution of the Dirichlet problem on the unit disk. For example, from the fact that every integrable real function has a unique corresponding inner analytic function we may conclude that the solution of the Dirichlet problem on the unit disk exists and is unique for any boundary condition $f(\theta)$ that is simply integrable on the unit circle. This extends the validity of the usual theorem to functions $f(\theta)$ that not only can be discontinuous, but that can also have integrable divergences at a finite number of singular points.

Since we have extended the set of inner analytic functions within the open unit disk to include on the unit circle a class of locally non-integrable real functions, the validity of the theorem is also extended to that class of singular boundary conditions, with a suitable reinterpretation of the set of boundary conditions. Finally, since we have extended the set of inner analytic functions within the open unit disk to include on the unit circle a class of singular generalized functions, the validity of the theorem is extended to that class of singular boundary conditions as well, with another suitable reinterpretation of the set of boundary conditions. Given the involvement of the Dirac delta "function", this is clearly related to the Green's-function approach to the solution of partial differential equations.

6 The Role of Integral-Differential Chains

In [2] we introduced the concept of an integral-differential chain of functions. This was done for the analysis of the convergence issues of the associated series on the unit circle. Associated to this, in that same paper we introduced the concepts of soft and hard singularities, as well as a corresponding gradation, in the form of the definition of a degree of softness or of a degree of hardness of any given soft or hard singularity. Let us now review the concept of an integral-differential chain in the context of our new criterion for defining inner analytic functions.

As we discussed in Section 2, given any integrable zero-average real function $f(\theta)$, we can construct from it an inner analytic function $w(z)$ such that the real function is recovered from its real part in the $\rho \to 1$ limit. Now, once we have this complex function, since it is analytic within the open unit disk, and hence both infinitely differentiable and infinitely integrable there, we can consider its successive logarithmic derivatives and logarithmic primitives, all of which exist and are equally analytic, in exactly the same domain. Both these operations were defined in [2], the logarithmic derivative as

$$w^{\cdot}(z) = z\,\frac{dw(z)}{dz},$$

which also gives our notation for it, and the logarithmic primitive as

$$w^{-1\cdot}(z) = \int_0^z dz'\,\frac{1}{z'}\,w(z'),$$

over any simple integration contour contained within the open unit disk, going from 0 to z. These two operations are the inverses of one another, as shown in [2]. Note that the operation of logarithmic integration was defined in a way that eliminates the usual arbitrary integration constant, and that both definitions keep invariant the property that $w(0) = 0$. In other words, both the logarithmic derivative and the logarithmic primitive of an inner analytic function are also inner analytic functions, according to our new definition.

We are thus able to define a discrete infinite chain of inner analytic functions, passing through the inner analytic function we started with, and running by logarithmic differentiation in one direction and by logarithmic integration in the other, indefinitely in both directions. Since the operations of logarithmic differentiation and logarithmic integration always produce definite and unique results, every zero-average integrable real function belongs to a single one of these chains. Other ways to state this are to say that two different chains cannot have a common element, or that two chains cannot "cross". They constitute a type of "discrete fibration" of the space of inner analytic functions. Also, since the operations of logarithmic differentiation and logarithmic integration correspond respectively to differentiation and integration with respect to θ on the unit circle, given that we have on any circle with radius $\rho \leq 1$ centered at the origin, as shown in in [2],

$$\begin{aligned} \frac{dw(z)}{d\theta} &= \imath w^{\cdot}(z), \\ \int_{z_0}^{z} d\theta'\, w(z') &= -\imath \left[w^{-1\cdot}(z) - w^{-1\cdot}(z_0) \right], \end{aligned}$$

it follows that in the $\rho \to 1$ limit the real part of each one of the inner analytic functions within a chain corresponds either to a derivative $f^{n\prime}(\theta)$ or to a primitive $f^{-n\prime}(\theta)$ of the original zero-average real function $f(\theta)$. We conclude therefore that, given an integral-differential chain of inner analytic functions, a corresponding chain of real objects on the unit circle is also defined. As we discussed in Section 3, these objects may or may not be integrable zero-average real functions. Whatever the case may be, we will from now on consider these objects to be an integral part of the chain, forming its real sector, as illustrated by the diagram that follows:

$$\begin{array}{ccccccccc} & & \ldots & f^{-2\prime}(\theta) & f^{-1\prime}(\theta) & f(\theta) & f^{1\prime}(\theta) & f^{2\prime}(\theta) & \ldots & & \\ \text{integration} & \leftarrow & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \rightarrow & \text{differentiation} \\ & & \ldots & w^{-2\cdot}(z) & w^{-1\cdot}(z) & w(z) & w^{1\cdot}(z) & w^{2\cdot}(z) & \ldots & . & \end{array}$$

Note that there may be singular points of the inner analytic functions on the unit circle, and that, as we discussed in [2], all the inner analytic functions within a chain have exactly the same set of singular points. Since any analytic function is both differentiable and integrable, and its derivative and primitive are equally analytic, in exactly the same domain as the original function, no point of singularity ever appears or vanishes in processes of differentiation and integration. Only the nature of the singular points, namely their degrees of softness or degrees of hardness, changes due to the operations of logarithmic differentiation and logarithmic integration.

In order to simplify the discussion that follows, let us establish some limitations on the singularities that may be present. First, in the spirit of the limitation on the inner analytic functions expressed by Equation (4), let us eliminate from this discussion the cases in which there are infinitely hard singularities, such as essential singularities. Second, if the inner analytic functions in a chain have any removable singularities, in what follows we will assume that they have been removed. We will also treat separately any infinitely soft singularities that may eventually exist in a chain. We are thus left only with singularities at which some real function in the chain is differentiable while its derivative is not. Finally, for simplicity of argumentation, let us limit the following discussion to the case in which there is a finite number of such singularities, which are then all isolated singularities.

We may easily establish a general classification of all the integral-differential chains, in a very simple way, by noting that if one of the real objects in a chain is a C^∞ zero-average real function, then all the real objects in that chain must also be C^∞ zero-average real functions. On the other hand, if one of the real objects is *not* a C^∞ zero-average real function, then none of them can be C^∞. Therefore, either a chain contains only C^∞ zero-average real functions on its real sector, or it contains none. We will call a chain that contains only C^∞ zero-average real functions a C^∞ chain. Note that the inner analytic functions in any chain are all, of course, C^∞ within the open unit disk, in the complex sense, and that this statement and the associated classification refers only to the real functions obtained on the unit circle, and to whether or not they are C^∞ on that circle, in the real sense. As was discussed in [2], the case of C^∞ chains is that in which the inner analytic functions have no singularities at all on the unit circle, or have only infinitely soft singularities, such as the one exemplified in one of the appendices of [2].

The other class of chains, which is the complement of the class of C^∞ chains, is the class of chains that include at least one singularity of the inner analytic functions over the unit circle, which is not an infinitely soft singularity. Let us recall that we assume that all removable singularities have been eliminated, so that there are none of those around to cloud the issue. Given the limitations on the types of singularities that we have established, it follows that this is a singularity at which, as we travel along the chain in the differentiation direction, we eventually pass from a differentiable zero-average real function to one that is not differentiable. From this fact one can draw two conclusions, if we start from the last differentiable real function in the chain. On the one hand, it follows that in the integration direction of the chain all subsequent real objects are also differentiable zero-average real functions. On the other hand, however, it also follows that in the differentiation direction we immediately encounter a zero-average real function that has a point of non-differentiability. Immediately after that, we will encounter a zero-average real function that has, at that singular point, either a step-type discontinuity or a divergence to infinity.

Let us consider first the case in which there is a divergence to infinity. If there is a singular point where the resulting real function diverges to infinity, then upon further differentiation the function will become non-integrable around that point. For example, if the divergence around the point θ_1 behaves as $1/(\theta-\theta_1)^p$ with $0 < p < 1$, which is integrable, then the first derivative behaves as $1/(\theta-\theta_1)^{p'}$ with $1 < p' < 2$, which is not integrable. Therefore, we see that there are at least some non-integrable real functions, in the sense that the integrals of their absolute values over the unit circle diverge to infinity, which are still representable by inner analytic functions. These are the locally non-integrable functions discussed in Section 4. In such cases the Fourier coefficients of the real function cannot be calculated directly on the unit circle in the usual way, but the functions can still be defined by means of the corresponding inner analytic function. These are the cases to which the definition of the extended Fourier coefficients can be applied.

Let us consider now the case in which there is a step-type discontinuity. Since the differentiation of a step-type discontinuity results in a delta "function", the next step in the differentiation direction will produce, therefore, a combination of real objects which includes a delta "function" at that point. After that, further steps in the differentiation direction will produce combinations of real objects which include the delta "function" and its successive higher-order derivatives. Therefore, every chain included in this sub-class contains as part of its real sector the delta "function" and *all* its multiple derivatives. We may conclude, therefore, that the set of singular real objects examined in Section 3, which is a subset of the set of all generalized functions, is an inevitable companion of the set of zero-average integrable real functions on the unit circle.

Both the zero-average integrable real functions and the set of radically singular objects represented by the delta "function" and its derivatives of all orders are inevitably integrated as part of one and the same structure. In addition to this, all locally non-integrable real functions that are derivatives of any finite order of integrable real functions are also part of that structure. In this way we see that the integral-differential chains show that all the real objects studied in this paper, either normal or generalized functions, either regular or singular, are closely integrated into a single overall structure.

7 Conclusions

We have shown that the correspondence between real functions and inner analytic functions established in earlier papers can be greatly extended. With a set of inner analytic functions that is still limited by the restriction expressed in Equations (2) and (4), adopted in order to avoid the most severe singularities at the unit circle, it is possible to define a much larger set of objects on the unit circle than was previously thought. This includes not only all integrable real functions, regardless of any parity properties, but at least some radically singular objects such as generalized functions, in the general spirit of the Schwartz theory of distributions, as well as at least a fairly large class of plainly non-integrable real functions.

This development lead to the introduction of an extended definition of the Taylor-Fourier coefficients of a real function, that can be used even for the class of non-integrable real functions just mentioned. Coupled with the technique of singularity factorization, which we introduced before, these extended coefficients have an algorithmic value in terms of the representation of these singular functions by trigonometric series. In a somewhat surprising way, the concept of integral-differential chains, introduced before for a completely different reason, found use here as a way to systematize at least a large part of this large set of real objects. In fact, it was instrumental for the very realization that it was possible to extend the structure to non-integrable real functions, and eventually led to the definition of the extended Taylor-Fourier coefficients given in Equation (12).

The set of inner analytic functions, as defined in the new standard way proposed in this paper, has the properties of a vector space with real scalars, in such a way that the addition of any two real objects on the unit circle always corresponds to the addition of the two corresponding inner analytic functions. In this way, arbitrary linear combinations of any of the real objects discussed here also correspond to inner analytic functions. This includes linear combinations mixing normal integrable functions, locally non-integrable functions and singular generalized functions. They can all be treated on the same footing. As a simple example, generalized functions with several singular points can be obtained by just adding the inner analytic functions corresponding to several generalized functions with one singular point each.

Also in a somewhat surprising turn of events, an interesting connection with the Dirichlet problem on the unit disk suggested itself in a rather forceful way. This led at once to simple and easy extensions of the basic theorems of existence and uniqueness of solutions to that problem, as well as to the proposition of new types of boundary value problems, with what we named punctured boundary conditions, which allow for the presence of almost arbitrary singular points at the boundary. One finds that the Dirichlet problem on the unit circle has a unique solution for any integrable boundary condition, which can be discontinuous and even unbounded at a finite set of singular points. The same is true for singular boundary conditions consisting of generalized functions or locally non-integrable functions.

The problem of the complete characterization of all the real objects on the unit circle corresponding to the set of inner analytic functions considered here remains open. One interesting question is whether or not all Lebesgue-measurable but non-integrable real functions are included in this structure, as limits of inner analytic functions to the unit circle. Another is whether or not there are other radically singular objects, such as but other than the Dirac delta "function" and its derivatives of all orders, that can also be found as part of the structure.

A Equivalence of Integrability and Local Integrability

Consider the set of Lebesgue-measurable real functions $f(\theta)$ defined on $[-\pi, \pi]$. Let us recall that within this set the conditions of integrability and of absolute integrability are equivalent conditions. The condition of local integrability is defined as integrability on all closed sub-intervals of $[-\pi, \pi]$. Since this includes $[-\pi, \pi]$ itself, it is immediate that the condition of local integrability implies the condition of integrability.

Now consider an arbitrary closed sub-interval $[a, b]$ of $[-\pi, \pi]$. Since the condition of integrability implies the condition of absolute integrability, if $f(\theta)$ is integrable on $[-\pi, \pi]$, then it is absolutely integrable there. But according to the properties of the operation of integration we have that

$$\int_{-\pi}^{\pi} d\theta\, |f(\theta)| = \int_{-\pi}^{a} d\theta\, |f(\theta)| + \int_{a}^{b} d\theta\, |f(\theta)| + \int_{b}^{\pi} d\theta\, |f(\theta)|.$$

If $f(\theta)$ is an absolutely integrable function, then in this relation the number in the left-hand side is a finite positive real number, while the right-hand side is the sum of three positive real numbers. It then follows that these three real numbers must all be finite, and therefore that $f(\theta)$ is absolutely integrable on $[a, b]$, since the integral of its absolute value on that interval is a finite real number. Now, since the condition of absolute integrability implies the condition of integrability, it follows that $f(\theta)$ is integrable on $[a, b]$.

Therefore, since $f(\theta)$ is thus seen to be integrable on an arbitrary closed sub-interval $[a, b]$ of $[-\pi, \pi]$, it follows that it is integrable on *all* such closed sub-intervals, and is therefore locally integrable. The final conclusion is that the conditions of integrability and of local integrability are equivalent conditions in the space of all Lebesgue-measurable real functions defined within $[-\pi, \pi]$. Hence in this context all three conditions, that of local integrability, that of absolute integrability and that of integrability, are equivalent conditions.

B Calculation of some Inner Analytic Functions

In this section we will derive a simple algebraic recursion relation for all the inner analytic functions associated to the multiple derivatives of the delta "function". Starting from the inner analytic function that corresponds to the delta "function", given in Equation (5) of the text,

$$w(z, z_1) = \frac{1}{2\pi} - \frac{1}{\pi}\,\frac{z}{z - z_1},$$

and from the differential recursion relation in terms of logarithmic derivatives that relates each inner analytic function in the sequence to the next one,

$$w_{\delta^{(n+1)\prime}}(z, z_1) = \imath z\,\frac{d}{dz} w_{\delta^{n\prime}}(z, z_1),$$

we may in principle derive all the inner analytic functions in the sequence. Doing this one gets for the first few cases of the inner analytic functions, including the one for the original delta "function", for which we use the notation $w_{\delta 0'}(z, z_1) = w_\delta(z, z_1)$, the list of results that follows,

$$
\begin{aligned}
w_{\delta 0'}(z, z_1) &= \frac{1}{2\pi} - \frac{1}{\pi \imath^0} \frac{z}{z - z_1}, \\
w_{\delta 1'}(z, z_1) &= -\frac{1}{\pi \imath^1} \frac{z z_1}{(z - z_1)^2}, \\
w_{\delta 2'}(z, z_1) &= -\frac{1}{\pi \imath^2} \frac{z(z + z_1) z_1}{(z - z_1)^3}, \\
w_{\delta 3'}(z, z_1) &= -\frac{1}{\pi \imath^3} \frac{z\left(z^2 + 4 z z_1 + z_1^2\right) z_1}{(z - z_1)^4}, \\
w_{\delta 4'}(z, z_1) &= -\frac{1}{\pi \imath^4} \frac{z\left(z^3 + 11 z^2 z_1 + 11 z z_1^2 + z_1^3\right) z_1}{(z - z_1)^5}, \\
w_{\delta 5'}(z, z_1) &= -\frac{1}{\pi \imath^5} \frac{z\left(z^4 + 26 z^3 z_1 + 66 z^2 z_1^2 + 26 z z_1^3 + z_1^4\right) z_1}{(z - z_1)^6}.
\end{aligned}
$$

Instead of working out the details of these first few cases, let us construct inductively a simple algebraic recursion relation which gives the general form of these inner analytic functions. We may write, in terms of the variable $\chi = z/z_1$, for $n > 0$, that

$$
w_{\delta n'}(\chi) = -\frac{1}{\pi \imath^n} \frac{\chi P_{n-1}(\chi)}{(\chi - 1)^{n+1}},
$$

where $P_{n-1}(\chi)$ is a polynomial of order $n - 1$ on χ. The first few polynomials, for $n > 0$, are therefore given by

$$
\begin{aligned}
P_0(\chi) &= 1, \\
P_1(\chi) &= 1 + \chi, \\
P_2(\chi) &= 1 + 4\chi + \chi^2, \\
P_3(\chi) &= 1 + 11\chi + 11\chi^2 + \chi^3, \\
P_4(\chi) &= 1 + 26\chi + 66\chi^2 + 26\chi^3 + \chi^4.
\end{aligned}
$$

Since we have the differential recurrence relation for the inner analytic function $w_{\delta n'}(z, z_1)$, in terms of logarithmic derivatives,

$$
w_{\delta(n+1)'}(\chi) = \imath \chi \frac{d}{d\chi} w_{\delta n'}(\chi),
$$

we may derive a related differential recurrence relation for the polynomials,

$$
\begin{aligned}
-\frac{1}{\pi \imath^{n+1}} \frac{\chi P_n(\chi)}{(\chi - 1)^{n+2}} &= \imath \chi \frac{d}{d\chi} \left[-\frac{1}{\pi \imath^n} \frac{\chi P_{n-1}(\chi)}{(\chi - 1)^{n+1}} \right] \\
&= -\frac{\imath \chi}{\pi \imath^n} \left[\frac{P_{n-1}(\chi)}{(\chi - 1)^{n+1}} + \frac{\chi P'_{n-1}(\chi)}{(\chi - 1)^{n+1}} - \frac{(n+1)\chi P_{n-1}(\chi)}{(\chi - 1)^{n+2}} \right] \\
&= \frac{\chi}{\pi \imath^{n+1}} \frac{-(n\chi + 1) P_{n-1}(\chi) + \chi(\chi - 1) P'_{n-1}(\chi)}{(\chi - 1)^{n+2}},
\end{aligned}
$$

where $P'_n(\chi)$ is the derivative of $P_n(\chi)$ with respect to χ. We get therefore for the polynomials the differential recurrence relation

$$P_n(\chi) = (n\chi + 1)P_{n-1}(\chi) - \chi(\chi - 1)P'_{n-1}(\chi).$$

In order to solve this recurrence relation we write the polynomials explicitly,

$$\begin{aligned}
P_n(\chi) &= \sum_{i=0}^{n} A_{n,i}\chi^i \quad \Rightarrow \\
P_{n-1}(\chi) &= \sum_{i=0}^{n-1} A_{n-1,i}\chi^i \quad \Rightarrow \\
P'_{n-1}(\chi) &= \sum_{i=0}^{n-1} iA_{n-1,i}\chi^{i-1},
\end{aligned}$$

so that we have for the differential recurrence relation

$$\begin{aligned}
\sum_{i=0}^{n} A_{n,i}\chi^i &= n\chi P_{n-1}(\chi) + P_{n-1}(\chi) - \chi^2 P'_{n-1}(\chi) + \chi P'_{n-1}(\chi) \\
&= n\sum_{i=0}^{n-1} A_{n-1,i}\chi^{i+1} + \sum_{i=0}^{n-1} A_{n-1,i}\chi^i - \sum_{i=0}^{n-1} iA_{n-1,i}\chi^{i+1} + \sum_{i=0}^{n-1} iA_{n-1,i}\chi^i \\
&= n\sum_{i=1}^{n} A_{n-1,i-1}\chi^i - \sum_{i=1}^{n}(i-1)A_{n-1,i-1}\chi^i + \sum_{i=0}^{n-1} A_{n-1,i}\chi^i + \sum_{i=0}^{n-1} iA_{n-1,i}\chi^i.
\end{aligned}$$

If we now read out the special case $i = 0$ we simply get $A_{n,0} = A_{n-1,0}$, which is valid for all $n > 0$. Since $P_0(\chi) = 1$ implies that $A_{0,0} = 1$, we have $A_{n,0} = 1$ for all n. The special case $i = n$ gives the equally simple result $A_{n,n} = A_{n-1,n-1}$, which is also valid for all $n > 0$. Since $A_{0,0} = 1$, we have $A_{n,n} = 1$ for all n. For all the other values of i we have the ordinary algebraic recurrence relation

$$\sum_{i=1}^{n-1} A_{n,i}\chi^i = \sum_{i=1}^{n-1}[(i+1)A_{n-1,i} + (n-i+1)A_{n-1,i-1}]\chi^i,$$

so that we get the two-index algebraic recurrence relation for the coefficients of the polynomials, given in Equation (11) of the text,

$$A_{n,i} = (i+1)A_{n-1,i} + (n-i+1)A_{n-1,i-1},$$

for $0 < i < n$. Observe that this relation implies that all coefficients $A_{n,i}$ will be integers. The special cases examined separately before can now be merged back in if we agree that $A_{n,i} = 0$ for $i < 0$ and for $i > n$. In the way of illustration, we may write some particular cases of this recurrence relation, choosing particular values of i. For $i = 1$ and $i = n - 1$ we get, respectively

$$\begin{aligned}
A_{n,1} &= nA_{n-1,0} + 2A_{n-1,1}, \\
A_{n,n-1} &= nA_{n-1,n-1} + 2A_{n-1,n-2},
\end{aligned}$$

which are recurrence relations in terms of n alone. Alternatively, starting from the coefficient $A_{n,0} = 1$, which has the value shown for all n, and using the algebraic recurrence relation for a fixed value of n, we may derive all the $n+1$ coefficients $A_{n,i}$ of the polynomial $P_n(\chi)$. In this way we may construct the table of values of $A_{n,i}$ shown in Table 1 of the text.

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