Fourier Theory on the Complex Plane IV
Representability of Real Functions by their Fourier Coefficients

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Abstract

The results presented in this paper are refinements of some results presented in a previous paper. Three such refined results are presented. The first one relaxes one of the basic hypotheses assumed in the previous paper, and thus extends the results obtained there to a wider class of real functions. The other two relate to a closer examination of the issue of the representability of real functions by their Fourier coefficients. As was shown in the previous paper, in many cases one can recover the real function from its Fourier coefficients even if the corresponding Fourier series diverges almost everywhere. In such cases we say that the real function is still representable by its Fourier coefficients. Here we establish a very weak condition on the Fourier coefficients that ensures the representability of the function by those coefficients. In addition to this, we show that any real function that is absolutely integrable can be recovered almost everywhere from, and hence is representable by, its Fourier coefficients, regardless of whether or not its Fourier series converges. Interestingly, this also provides proof for a conjecture proposed in the previous paper.

1 Introduction

In a previous paper [1] we have developed a correspondence between, on one side, the Fourier series and Fourier coefficients of real functions on the interval $[-\pi, \pi]$ and, on the other side, a complex analytic structure within the open unit disk, consisting of a set of inner analytic functions and their complex Taylor series. The reader is referred to that paper for the definition of many of the concepts and notations used in this one. In many cases this correspondence leads to the formulation of expressions involving modified trigonometric series that can converge very fast to a given real function, even when the Fourier series of that function diverges or converges very slowly, as shown in [2].

In order to establish the correspondence described above, in [1] we assumed that the Definite Parity (DP) real functions $f(\theta)$ under examination are such that their Fourier series and the corresponding Fourier Conjugate (FC) series converge together on at least one point on the interval $[-\pi, \pi]$, a domain which, in the context of the correspondence to be established, is mapped onto the unit circle of the complex plane. In this paper we will show that one can relax that hypothesis, replacing it by a much weaker one.

Following the development in [1], we will deal here only with real functions that have definite parity properties, which we will call Definite Parity real functions or DP real functions for short. Since any real function $f(\theta)$ in the interval $[-\pi,\pi]$, without any restriction, can be separated into even and odd parts,

$$
\begin{aligned}
f(\theta) &= f_{\rm c}(\theta)+f_{\rm s}(\theta),\\
f_{\rm c}(\theta) &= \frac{f(\theta)+f(-\theta)}{2},\\
f_{\rm s}(\theta) &= \frac{f(\theta)-f(-\theta)}{2},
\end{aligned}
$$

where

$$
\begin{aligned}
f_{\rm c}(\theta) &= f_{\rm c}(-\theta),\\
f_{\rm s}(\theta) &= -f_{\rm s}(-\theta),
\end{aligned}
$$

we can restrict the discussion to Definite Parity (DP) real functions without any loss of generality. This condition implies that the corresponding inner analytic function $w(z)$ has the property that it reduces to a real function on the interval $(-1,1)$ of the real axis, as discussed in [1]. For simplicity, we will also assume that $f(\theta)$ is a zero-average function, since adding a constant function to $f(\theta)$ is a trivial operation that does not significantly affect the issues under discussion here. This condition implies that the corresponding inner analytic function $w(z)$ has the property that $w(0)=0$, as discussed in [1].

In Subsection 8.1 of [1] we have shown that, if there is any singularity of an analytic function $w(z)$ inside the open unit disk, then the sequence of Taylor-Fourier coefficients of its Taylor series diverges to infinity exponentially fast on the unit circle, as a function of the series index. In this paper we will discuss the converse of this statement. We will show here that the mere absence of an exponentially-fast divergence of the sequence of Fourier coefficients on the unit circle is enough to ensure the convergence of the corresponding complex power series inside the open unit disk, thus leading to the definition of a corresponding inner analytic function.

A note about the concept of integrability of real functions is in order at this point. What we mean by integrability in this paper is integrability in the sense of Lebesgue, with the use of the usual Lebesgue measure. We will assume that all the real functions at issue here are measurable in the Lebesgue measure. Therefore whenever we speak of real functions, it should be understood that we mean Lebesgue-measurable real functions. We will use the following result from the theory of measure and integration: for real functions defined within a compact interval, which are Lebesgue measurable, integrability and absolute integrability are equivalent conditions [3]. Therefore we will use the concepts of integrability and of absolute integrability interchangeably.

2 Weakening the Convergence Hypothesis

Let $f(\theta)$ be a DP real function that has zero average value, with $\theta\in[-\pi,\pi]$. We will assume that $f(\theta)$ is an integrable function, so that its Fourier coefficients α_k and β_k exist, since they are given by

$$
\begin{aligned}
\alpha_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, f(\theta)\cos(k\theta),\\
\beta_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, f(\theta)\sin(k\theta).
\end{aligned}
$$

Note that $\alpha_0 = 0$ because $f(\theta)$ is assumed to be a zero-average function. We will name such coefficients a_k, for $k = 1, 2, 3, \ldots, \infty$, irrespective of whether $f(\theta)$ is even or odd, and thus of whether it gives origin respectively to a cosine or sine series. Let us assume that the a_k coefficients satisfy the following hypothesis: given a sequence a_k of coefficients, there exists a positive real function $g(k)$ with the property that

$$\lim_{k\to\infty} g(k) = 1, \tag{1}$$

and such that for all $k \geq 1$

$$\left| \frac{a_{k+1}}{a_k} \right| \leq g(k). \tag{2}$$

What this means is that we assume that the ratio of the absolute values of two consecutive coefficients is bounded from above by a function that tends to one in the $k \to \infty$ limit. Note that the $k \to \infty$ limit of the ratio of coefficients itself may not even exist. Note also that, if this condition holds for a given DP real function, then it automatically holds for the corresponding FC real function as well, since both have the same coefficients and the condition is imposed only on these coefficients. Following the development described in [1], given $f(\theta)$ with such properties we may now construct the two DP trigonometric series

$$\begin{aligned} S_{\rm c} &= \sum_{k=1}^{\infty} a_k \cos(k\theta), \\ S_{\rm s} &= \sum_{k=1}^{\infty} a_k \sin(k\theta), \end{aligned}$$

which are the FC series of one another, and then the complex power series

$$\begin{aligned} S_z &= \sum_{k=1}^{\infty} a_k \rho^k \left[\cos(k\theta) + \imath \sin(k\theta)\right] \\ &= \sum_{k=1}^{\infty} a_k z^k, \end{aligned}$$

where $z = \rho \exp(\imath\theta)$, with $\rho \geq 0$, so that $S_{\rm c}$ and $S_{\rm s}$ are respectively the real and imaginary parts of S_z for $\rho = 1$. Note that the condition expressed by Equations (1) and (2) does not, by itself, imply the convergence of the series $S_{\rm c}$ and $S_{\rm s}$ on the unit circle, since sequences of coefficients a_k that do not go to zero as $k \to \infty$ may satisfy it. Therefore, many sequences of coefficients leading to Fourier series that diverge on the unit circle satisfy that condition. If we now use the ratio criterion to analyze the convergence of the power series S_z inside the open unit disk, we get

$$\left| \frac{a_{k+1} z^{k+1}}{a_k z^k} \right| = \left| \frac{a_{k+1}}{a_k} \right| \rho.$$

Our hypothesis about the ratio of the a_k coefficients now leads to the relation, inside the open unit disk,

$$\left| \frac{a_{k+1} z^{k+1}}{a_k z^k} \right| \leq \rho g(k).$$

Now, given any value of $\rho < 1$, since the limit of $g(k)$ for $k \to \infty$ is one, it follows that the same limit of $\rho g(k)$ is strictly less than one. Therefore, we may conclude that there is a value k_m of k such that, if $k > k_m$, then $\rho g(k) < 1$, so that the ratio is less that one, thus implying that we have

$$\left| \frac{a_{k+1} z^{k+1}}{a_k z^k} \right| < 1,$$

for $k > k_m$. This implies that the ratio criterion is satisfied and therefore that the series S_z converges at such points. Since this is true for any $\rho < 1$, we may conclude that the power series S_z converges on the open unit disk. Therefore, the series converges to a complex analytic function $w(z)$ there, which is an inner analytic function according to the definition given in [1].

The results established in [1] then imply that $f(\theta)$ can now be obtained almost everywhere over the unit circle as the $\rho \to 1$ limit of the real or imaginary part of $w(z)$, as the case may be. Note that the series S_z, and hence the series S_c and S_s, may still be divergent over the whole unit circle. This does not affect the recovery of the real function from its Fourier coefficients in the manner just described. We have therefore shown that the correspondence established in [1] holds without the hypothesis that S_c and S_s be convergent together on at least at one point of the unit circle, so long as the Taylor-Fourier coefficients a_k satisfy the weaker condition expressed by Equations (1) and (2).

3 Representability by the Fourier Coefficients

Again we start with an arbitrary integrable DP real function $f(\theta)$ that has zero average value, with $\theta \in [-\pi, \pi]$. Let us now assume that this function is such that the corresponding Fourier coefficients a_k satisfy the condition that

$$\lim_{k\to\infty} |a_k| \,\mathrm{e}^{-Ck} = 0, \tag{3}$$

for all real $C > 0$. What this means is that a_k may or may not go to zero as $k \to \infty$, may approach a non-zero real number, and may even diverge to infinity as $k \to \infty$, so long as it does not do so exponentially fast. This includes therefore not only the sequences of Fourier coefficients corresponding to all possible convergent Fourier series, but many sequences that correspond to Fourier series that diverge almost everywhere. In fact, it even includes sequences of coefficients that cannot be obtained at all from a real function, such as the sequence $a_k = 1/\pi$ for all $k \geq 1$, which is associated to the Dirac delta "function" $\delta(\theta)$, as shown in [1] and as will be discussed in Section 5 of this paper. It is therefore a very weak condition indeed.

Before anything else, let us establish a preliminary result, namely that the condition in Equation (3) implies that we also have

$$\lim_{k\to\infty} |a_k| k^p \,\mathrm{e}^{-Ck} = 0, \tag{4}$$

for all real powers $p > 0$. This is just a formalization of the well-known fact that the negative-exponent real exponential function goes to zero faster than any positive power goes to infinity, as $k \to \infty$. We may write the function of k on the left-hand side as

$$|a_k| k^p \,\mathrm{e}^{-Ck} = |a_k| \,\mathrm{e}^{p \ln(k)} \,\mathrm{e}^{-Ck}.$$

Recalling the properties of the logarithm, we now observe that, given an arbitrary real number $A > 0$, there is always a sufficiently large value k_m of k above which $\ln(k) < Ak$. A simple proof can be found in Appendix A. Due to this we may write, for all $k > k_m$,

$$|a_k|k^p\,\mathrm{e}^{-Ck} < |a_k|\,\mathrm{e}^{pAk}\,\mathrm{e}^{-Ck},$$

since the exponential is a monotonically increasing function. If we choose $A = C/(2p)$, which is positive and not zero, we get that, for all $k > k_m$,

$$\begin{aligned} |a_k|k^p\,\mathrm{e}^{-Ck} &< |a_k|\,\mathrm{e}^{Ck/2}\,\mathrm{e}^{-Ck} \\ &= |a_k|\,\mathrm{e}^{-Ck/2}. \end{aligned}$$

According to our hypothesis about the coefficients a_k, the $k \to \infty$ limit of the expression in the right-hand side is zero for any strictly positive value of $C' = C/2$, so that taking the $k \to \infty$ limit we establish our preliminary result,

$$\lim_{k\to\infty} |a_k|k^p\,\mathrm{e}^{-Ck} = 0,$$

for all real $C > 0$ and all real $p > 0$. If we now construct the complex power series S_z as before, using the coefficients a_k, we are in a position to show that it is absolutely convergent inside the open unit disk. In order to do this we consider the real power series $\overline{S}_z$ of the absolute values of the terms of the series S_z, which we write as

$$\begin{aligned} \overline{S}_z &= \sum_{k=1}^{\infty} |a_k|\rho^k \\ &= \sum_{k=1}^{\infty} |a_k|\,\mathrm{e}^{k\ln(\rho)}. \end{aligned}$$

Since $\rho < 1$ inside the open unit disk, the logarithm shown is strictly negative, and we may put $\ln(\rho) = -C$ with real $C > 0$. We can now see that, according to our hypothesis about the coefficients a_k, the terms of this series go to zero exponentially fast as $k \to \infty$. This already suffices to establish its convergence, but we may easily make this more explicit, writing

$$\begin{aligned} \overline{S}_z &= \sum_{k=1}^{\infty} |a_k|\,\mathrm{e}^{-Ck} \\ &= \sum_{k=1}^{\infty} \frac{k^2|a_k|\,\mathrm{e}^{-Ck}}{k^2}. \end{aligned}$$

According to our preliminary result in Equation (4) the numerator shown goes to zero as $k \to \infty$, and therefore above a sufficiently large value k_m of k it is less that one, so we may write that

$$\begin{aligned} \overline{S}_z &= \sum_{k=1}^{k_m} |a_k|\,\mathrm{e}^{-Ck} + \sum_{k=k_m+1}^{\infty} \frac{k^2|a_k|\,\mathrm{e}^{-Ck}}{k^2} \\ &< \sum_{k=1}^{k_m} |a_k|\,\mathrm{e}^{-Ck} + \sum_{k=k_m+1}^{\infty} \frac{1}{k^2}. \end{aligned}$$

The first term on the right-hand side is a finite sum and therefore is finite, and the second term can be bounded from above by a convergent asymptotic integral on k, so that we have

$$
\begin{aligned}
\overline{S}_z &< \sum_{k=1}^{k_m} |a_k| \, \mathrm{e}^{-Ck} + \int_{k_m}^{\infty} dk \, \frac{1}{k^2} \\
&= \sum_{k=1}^{k_m} |a_k| \, \mathrm{e}^{-Ck} + \frac{-1}{k} \left[{}_{k_m}^{\infty} \right. \\
&= \sum_{k=1}^{k_m} |a_k| \, \mathrm{e}^{-Ck} + \frac{1}{k_m}.
\end{aligned}
$$

It follows that $\overline{S}_z$, which is a real sum of positive terms, so that its partial sums form a monotonically increasing sequence, is bounded from above and is therefore convergent. It then follows that S_z is absolutely convergent and therefore convergent. Since this is valid for all $\rho < 1$, we may conclude that S_z converges on the open unit disk. We may now recover $f(\theta)$ as the $\rho \to 1$ limit of the real or imaginary part of $w(z)$, as the case may be, almost everywhere on the unit circle, as was mentioned before and shown in [1].

This provides therefore a very general condition on the Fourier coefficients of the real functions that ensures that the correspondence established in [1] holds. This then ensures that the real functions can be recovered from, and therefore can be represented by, their Fourier coefficients. Note that the condition in Equation (3) can be considered as an even weaker form of the condition discussed in the previous section. We are therefore ready to state our first important conclusion:

If the sequence of Fourier coefficients a_k of a DP real function $f(\theta)$ satisfies the weak condition in Equation (3), so that the coefficients do not diverge to infinity exponentially fast or faster when $k \to \infty$, then the complex power series constructed from them converges to an inner analytic function inside the open unit disk, from which the real function can be recovered almost everywhere.

Note that, although we formulated this condition in terms of the Fourier coefficients a_k of a given DP real function, the fact that a_k are the Fourier coefficients of the function has in fact not been used at all. Therefore, the conclusion is valid for any sequence of coefficients that satisfies Equation (3), regardless of whether or not they can be obtained as the Fourier coefficients of some real function.

4 Analytic Criterion for Real Functions

Finally, let us establish a simple analytic condition over the real functions that ensures that they are representable by their Fourier coefficients. If we assume that $f(\theta)$ is absolutely integrable, so that the integral

$$
\frac{1}{\pi} \int_{-\pi}^{\pi} d\theta \, |f(\theta)| = F
$$

exists and is a finite real number F, then it follows that we have for the Fourier coefficients, taking as an example the case of the cosine series,

$$
\begin{aligned}
|a_k| &= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta \, f(\theta) \cos(k\theta) \right| \\
&\leq \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta \, |f(\theta)| |\cos(k\theta)|
\end{aligned}
$$

$$\begin{aligned} &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\,|f(\theta)| \\ &= F, \end{aligned}$$

where we used the triangular inequalities. It follows that we have, for all $k \geq 1$,

$$|a_k| \leq F.$$

Since we thus see that the Fourier coefficients of $f(\theta)$ are bounded within the interval $[-F, F]$, for all $k \geq 1$, it follows that they cannot diverge to infinity as $k \to \infty$, and therefore that they satisfy our hypothesis in Equation (3), namely that

$$\lim_{k\to\infty} |a_k|\, \mathrm{e}^{-Ck} = 0,$$

for all real $C > 0$. The same result can be established in a similar way for the case of the sine series, of course. It therefore follows that $f(\theta)$ is representable by its Fourier coefficients.

If we go back to a more general function $f(\theta)$ that has both even and odd parts, since the result holds for both parts, since we may also add a constant term without changing the result, and since we have limited ourselves to Lebesgue-measurable real functions within a compact interval, for which integrability and absolute integrability are one and the same concept, we are ready to state our second important conclusion:

Any integrable real function $f(\theta)$ defined within $[-\pi, \pi]$ is representable by its Fourier coefficients, and can be recovered from them almost everywhere in that interval, regardless of whether or not the corresponding Fourier series converges.

It is an interesting observation that this provides an answer to the conjecture proposed in [1], about whether or not there are any integrable real functions such that their sequences of Fourier coefficients a_k give rise to complex power series S_z which are strongly divergent, that is, that have at least one singular point strictly within the open unit disk. The answer, according to the proof worked out here, is that there are none, as expected.

5 Discussion and Extensions

In their use of Fourier series for the solution of physical problems, physicists often, and quite successfully, simply ignore convergence issues of the Fourier series involved. With just a bit of common sense and a willingness to accept approximate results, they just plow ahead with their calculations, and that seldom leads them into serious trouble. This is true even if one includes the occasional appearance in these calculations of the Fourier series of singular objects such as Dirac's delta "function", since there is usually not much difficulty in interpreting the divergences in physical terms.

The results established here can be viewed as an explanation of this rather remarkable fact. Functions that are useful in physics applications are always at the very least Lebesgue-measurable, and most often at least locally integrable almost everywhere. Extremely pathological functions are not of any use in such circumstances. Therefore, for all real functions of interest in physics applications, there is in effect an underlying analytic structure that firmly anchors all the operations which are performed on the Fourier series, mapping them onto corresponding and much safer operations on the inner-analytic functions within the open unit disk. These operations may include anything from the basic arithmetic operations to integration and differentiation, and so on.

The representation of real functions by their Fourier coefficients, which in physics usually goes by the name of “representation in momentum space”, often can be interpreted directly in terms of physical concepts. In fact, this alternative representation is frequently found to be the more important and fundamental one. In terms of the mathematical structure that we are dealing with here, it is clear that this state of affairs relates closely to the fact that the analytic structure within the open unit disk is quite clearly the more fundamental aspect of this whole mathematical structure. In the usual physics parlance, that analytic structure is an exact and universal “regulator” for all integrable real functions.

It may be possible to further extend the result presented in the previous section. Observe that if the real function $f(\theta)$ is limited and integrable, then its Fourier coefficients a_k are also limited, and thus it is obvious that they satisfy the condition in Equation (3), so that they do not diverge exponentially fast with k. However, the function does not have to be limited in order for the coefficients to satisfy that condition. The function may diverge to infinity at an isolated point, so long as the asymptotic integral around that point exists. If the function diverges to infinity at a point θ_0 in such a way that the integrals

$$\begin{aligned} I_{\ominus} &= \int_{\theta_{\ominus}}^{\theta_0} d\theta\, f(\theta), \\ I_{\oplus} &= \int_{\theta_0}^{\theta_{\oplus}} d\theta\, f(\theta), \end{aligned}$$

exist and are finite for some $\theta_{\ominus}$ and $\theta_{\oplus}$, with $\theta_{\ominus} < \theta_0 < \theta_{\oplus}$, and where there are no other hard singularities of $f(\theta)$ within the intervals $[\theta_{\ominus}, \theta_0)$ and $(\theta_0, \theta_{\oplus}]$, then the function is integrable and thus representable by its Fourier coefficients. One may also have a finite number of such isolated points without disturbing these properties. One may even consider the inclusion of a denumerable infinity of such points, so long as the numerical series resulting from the sum of the contributions of all the singular points to the integral is absolutely convergent, since otherwise the function cannot be considered to be integrable.

Another interesting extension of the results presented in the two previous sections would be one leading of the inclusion in the structure of singular objects which are not real functions, such as extended functions or distributions in the sense of Schwartz [4], represented by their distribution kernels, such as Dirac’s delta “function”. This extension seems to be relatively straightforward, and these objects are routinely dealt with, without too much trouble, in physics applications. These singular objects are also associated to inner analytic functions.

Note that, since the Fourier coefficients of any absolutely integrable real function are necessarily limited, these sequences of coefficients form only a small subset of the set of all the sequences of coefficients that satisfy the condition stated in Equation (3), since many sequences of coefficients which are not limited may satisfy that condition. This shows once again that the condition on the coefficients is in fact very weak. Besides, that condition may be applied to any sequence of real coefficients, whether or not they are the Fourier coefficients of a real function. Therefore, the condition includes much more than just real functions, since there may be many sequences of coefficients a_k that satisfy it but that are not obtainable from a real function on the unit circle, as its sequence of Fourier coefficients.

In other words, there are inner analytic functions within the open unit disk that correspond to definite sequences of coefficients a_k satisfying our hypothesis, but that are not related to a real function on the unit circle. These inner analytic functions have at least one hard singularity over the unit circle, such as a simple pole, or harder. A hard singularity is defined in [2] as a point at which the limit of the inner-analytic function does not exist, or diverges to infinity. A related gradation in terms of degrees of hardness is also defined there. One important example of this is the inner analytic function associated to the Dirac delta "function", which was given in [1]. Given a point z_1 on the unit circle, the very simple analytic function

$$w_\delta(z) = \frac{1}{2\pi} - \frac{1}{\pi}\,\frac{z}{z - z_1},$$

which has a simple pole on the unit circle, is an extended inner analytic function, that is, an inner analytic function rotated by the angle θ_1 associated to $z_1 = \exp(\imath\theta_1)$ and with the constant shown added to it. This analytic function within the open unit disk is such that the delta "function" can be obtained as the limit to the unit circle of its real part,

$$\delta(\theta - \theta_1) = \lim_{\rho \to 1} \Re[w_\delta(z)],$$

as was shown in [1] with basis on the properties that define the delta "function". It would be interesting to further investigate this extension, which would probably include every integrable object which is not a real function. In particular, it would be interesting to investigate whether or not there is a condition such as the condition of absolute integrability which is a sufficient condition, and possibly also a necessary condition, for the representability of such objects by their Fourier coefficients.

6 Conclusions

Two very weak conditions were established leading to the existence of corresponding inner analytic functions, and thus to the representability of real functions or other objects defined on the unit circle by their sequence of Fourier coefficients. One of them is a condition on the sequence of coefficients, the other is an analytic condition on the real functions. The first one, stated in the most general way possible, reads:

If a sequence of real coefficients a_k satisfies the weak condition in Equation (3), so that the coefficients do not diverge to infinity exponentially fast or faster when $k \to \infty$, then the complex power series constructed from them converges to an inner analytic function inside the open unit disk.

This is a very weak condition on the sequence of coefficients, leading to the representability of the object related to it by an inner analytic function. The object at issue may be an integrable real function, or it may be a singular object that has an integrability concept associated to it. The typical example of such singular objects is Dirac's delta "function" and the Schwartz distribution associated to it. This is therefore a very general condition, which in fact extrapolates the strict realm of real functions. The second one reads:

Any Lebesgue-measurable integrable real function $f(\theta)$ defined within $[-\pi, \pi]$ is representable by its Fourier coefficients, and can be recovered from them almost everywhere in that interval, regardless of whether or not the corresponding Fourier series converges.

This is a very weak analytical condition on the real functions, leading to the representability of each real function by a corresponding inner analytic function. Observe that, so long as the real functions at issue are Lebesgue-measurable, this is the statement that *every* integrable function is representable by its sequence of Fourier coefficients. All that is really needed is that these Fourier coefficients, and hence the integrals that give them, exist.

If one accepts the premise that the condition of integrability is necessary for the very existence of the sequence of Fourier coefficients, and since it is seen here to be also sufficient for the representability of the function by those coefficients, we may say that what we have here is in essence a necessary and sufficient condition for the representability of real functions by their Fourier coefficients, as defined by the usual integrals. This is similar in nature to the as yet open problem of establishing a necessary and sufficient condition for the convergence of the Fourier series of a real function. If we reinterpret this last one as a condition for the representability of the function by its series, we see that such a necessary and sufficient condition can be achieved if one exchanges the condition of representability by the series for a more general one, involving the representability by the sequence of coefficients, and the recovery of the real function from them via the construction of an analytic function within the open unit disk.

However, it is important to note that, although this condition is always sufficient, it is only necessary if one assumes that the Fourier coefficients must be given by the usual integrals over the periodic interval. As we will see elsewhere, there is in fact a large class of non-integrable real functions for which a set of Fourier coefficients can still be consistently defined, and which can still be represented by this set of Fourier coefficients, almost everywhere over the unit circle.

Although the process of recovery of the real function as a limit of the corresponding inner analytic function is, by itself, not algorithmic in nature, it can lead to algorithmic solutions in at least some cases. The results obtained here simply guarantee the existence of the inner analytic function, and at least in principle the possibility of the recovery of the real function through the limit of that inner analytic function to the unit circle. In sufficiently simple cases, namely when the inner analytic function has only a finite number of sufficiently soft known singularities on the unit circle, it is possible to devise expressions involving modified trigonometric series that converge to the function, as was shown in [2]. In this case, one acquires an algorithmic method for the calculation of the real function to any desired level of precision, and thus for the practical recovery of the real function from the Fourier coefficients, which can be made good enough for any practical purpose.

A A Property of the Logarithm

Let us show that the logarithm has the property that given an arbitrary real number $A > 0$, there is always a sufficiently large value k_m of the integer k above which $\ln(k) < Ak$. We simply promote k to a continuous real variable $x > 0$ and consider the function

$$h(x) = Ax - \ln(x).$$

Is is quite clear that this function diverges to positive infinity as $x \to 0$. If we calculate the first and second derivatives of $h(x)$ we get

$$\begin{aligned} h'(x) &= A - \frac{1}{x}, \\ h''(x) &= \frac{1}{x^2}. \end{aligned}$$

It is now clear that there is a single critical point where the first derivative is zero, that is where $h'(x_0) = 0$, given by $x_0 = 1/A$. At this point we have for the second derivative

$$h''(x_0) = A^2,$$

which is positive, implying that the critical point is a local minimum. Since there is no other minimum, maximum or inflection point, it becomes clear that the function must decrease from positive infinity as x increases from zero, go through the point of minimum at x_0, and then increase without limit as $x \to \infty$. At this point of minimum we have for the function itself,

$$h(x_0) = 1 + \ln(A).$$

It follows that, if $h(x_0) > 0$, then we must have $h(x) > 0$ for all $x > 0$. This corresponds to $A > 1/e$. On the other hand, if $h(x_0) \leq 0$, then there are two solutions x_1 and x_2 of the equation $h(x) = 0$, that coincide if $h(x_0) = 0$. This corresponds to $0 < A \leq 1/e$. In this case the function is positive within the interval $(0, x_1)$, negative within (x_1, x_2) and positive for $x > x_2$.

Therefore, for all possible values of A there is a value x_m of x, either $x_m = 0$ or $x_m = x_2$, such that for $x > x_m$ the function $h(x)$ is positive, and therefore

$$Ax > \ln(x).$$

Translating the statement back in terms of k, we have that given any real number $A > 0$, there is a minimum value k_m of the integer k such that $Ak > \ln(k)$. Thus the required statement is established.

B Proof of Uniqueness Almost Everywhere

Given a sequence of Fourier coefficients a_k of an integrable DP real function $f(\theta)$ with zero average, let us show that they uniquely characterize that real function almost everywhere, that is, up to a zero-measure function. The point of the proof is to show that this is true independently of the convergence of the Fourier series of the function, hence including the cases in which that series diverges almost everywhere. Since two real functions with opposite parities clearly must always have different sequences of Fourier coefficients, it is enough to show that the result holds within the two sets of real functions with the same parity. Although the definition of the integral in itself will not be used directly, the argument will involve the concept of a zero-measure function, and therefore we should always think in terms of the Lebesgue integral and of the usual Lebesgue measure, for conceptual reasons as well as for the sake of generality. Hence, when we talk here of an integrable function, we mean integrable in the sense of Lebesgue.

Imagine then that we are given two different integrable DP real functions $f_1(\theta)$ and $f_2(\theta)$, both with zero average and both with the same parity, and consider their difference

$$g(\theta) = f_1(\theta) - f_2(\theta),$$

which of course has the same parity as $f_1(\theta)$ and $f_2(\theta)$, and which is also a zero-average function. Let us assume that both $f_1(\theta)$ and $f_2(\theta)$ have the same sequence of Fourier coefficients a_k, for $k = 1, 2, 3, \ldots, \infty$. It follows that $g(\theta)$ has all its Fourier coefficients b_k equal to zero, since from the relation above we clearly have

$$b_k = a_k - a_k,$$

for all $k \geq 1$. This means that the function $g(\theta)$ has zero scalar product with all the elements of the Fourier basis, and thus that it is orthogonal to all of them. However, as shown in [1], that basis satisfies a completeness relation,

$$\delta(\theta - \theta_1) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \left[\cos(k\theta)\cos(k\theta_1) + \sin(k\theta)\sin(k\theta_1) \right], \tag{5}$$

for all θ and all θ_1, and is therefore complete. From this it follows that $g(\theta)$, being orthogonal to all the elements of the basis, must be a zero-measure function, that is, a function which is zero-measure equivalent to the identically zero function. In order to show this, we simply multiply the equation above by $g(\theta)$ and integrate on θ over the periodic interval,

$$\begin{aligned}
\int_{-\pi}^{\pi} d\theta\, g(\theta)\delta(\theta - \theta_1) &= \int_{-\pi}^{\pi} d\theta\, g(\theta)\, \frac{1}{2\pi} + \\
&\quad + \int_{-\pi}^{\pi} d\theta\, g(\theta)\, \frac{1}{\pi} \sum_{k=1}^{\infty} \left[\cos(k\theta)\cos(k\theta_1) + \sin(k\theta)\sin(k\theta_1) \right] \\
&= \frac{1}{2} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, g(\theta) \right] + \\
&\quad + \sum_{k=1}^{\infty} \left\{ \left[\frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, g(\theta)\, \cos(k\theta) \right] \cos(k\theta_1) \, + \right. \\
&\qquad \left. + \left[\frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, g(\theta)\, \sin(k\theta) \right] \sin(k\theta_1) \right\}.
\end{aligned}$$

The integral on the left-hand side can be calculated through the use of the properties of the delta "function", which were given and demonstrated in [1]. The three expressions within square brackets on the right-hand side are the various Fourier coefficients of the function $g(\theta)$. The first one is zero because $g(\theta)$, just like $f_1(\theta)$ and $f_2(\theta)$, is a zero-average function. The other two sequences of coefficients are identically zero because the scalar products of $g(\theta)$ and the elements of the basis are all zero. Besides, one of the sequences contains the coefficients of the part of the basis which has the wrong parity, and thus these coefficients must all be zero by a parity argument. Therefore the whole right-hand side is zero and hence we have

$$g(\theta_1) = 0,$$

which is valid for almost all values of θ_1. Therefore $g(\theta)$ is zero almost everywhere. In other words, $f_1(\theta)$ and $f_2(\theta)$ differ only by a zero-measure function, and are thus equal to each other almost everywhere.

We may conclude from this that, if $f_1(\theta)$ and $f_2(\theta)$ are to be different by more than a zero-measure function, then their sequences of Fourier coefficients must be different, and therefore the coefficients uniquely characterize the originating function, up to a zero-measure function. Observe that nothing in this argument involves the convergence of the Fourier series of the real functions. In particular, the proof of completeness of the Fourier basis presented in [1] and expressed by the relation in Equation (5) was obtained directly from the complex analytic structure within the open unit disk, and it also does not involve in any way the convergence of the Fourier series on the unit circle.

C Zero-Measure Equivalence Classes

The whole structure we are examining here induces one to think that it would be a reasonable thing to do if we decided to group all the real functions into zero-measure equivalence classes. We could consider as equivalent, at least from the point of view of the physics applications, two real functions which differ by a zero-measure function. For this purpose, a zero-measure real function is a function whose absolute integral over $[-\pi, \pi]$ is zero, or that has zero integral on any closed sub-interval of the interval $[-\pi, \pi]$.

Then one would consider in a group all functions that are zero measure in this sense, such as the identically null function. This class could then be represented simply by that particular element, $f(\theta) \equiv 0$, which is quite clearly the smoothest element within that class. Given any real function with non-zero measure, all other functions that differ from it by a zero-measure function would be in the same equivalence class. Instead of considering all real functions, we could formulate everything that was discussed here in terms of these equivalence classes.

Since the Fourier coefficients are defined by integrals, it is immediately clear that a given sequence of Fourier coefficients, if it belongs to any real function at all, belongs to one such equivalence class, rather than to the individual functions. Therefore, there is also some mathematical sense to such a classification. The proposed method of representation of the real functions, as limits to the unit circle of real or imaginary parts of inner analytic functions within the open unit disk, gives us then a clear definition of a preferred or central element of each class: that particular real function which is obtained at the limit just described. Being a limit from a perfectly smooth analytic function, this is clearly the smoothest element of the class.

One could consider this classification process, and the representation of each zero-measure equivalence class by its smoothest element, as a process of elimination of what we might call the irrelevant pathologies of the real functions. The only pathologies that would remain would be those that have a definite effect on the integral of the function. This certainly makes sense in terms of the physics applications, but it might be of some intrinsic mathematical interest as well.

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