Fourier Theory on the Complex Plane II
Weak Convergence, Classification and Factorization of Singularities

Jorge L. deLyra
Department of Mathematical Physics
Physics Institute
University of São Paulo

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Abstract

The convergence of DP Fourier series which are neither strongly convergent nor strongly divergent is discussed in terms of the Taylor series of the corresponding inner analytic functions. These are the cases in which the maximum disk of convergence of the Taylor series of the inner analytic function is the open unit disk. An essentially complete classification, in terms of the singularity structure of the corresponding inner analytic functions, of the modes of convergence of a large class of DP Fourier series, is established. Given a weakly convergent Fourier series of a DP real function, it is shown how to generate from it other expressions involving trigonometric series, that converge to that same function, but with much better convergence characteristics. This is done by a procedure of factoring out the singularities of the corresponding inner analytic function, and works even for divergent Fourier series. This can be interpreted as a resummation technique, which is firmly anchored by the underlying analytic structure.

1 Introduction

In this paper we will discuss the question of the convergence of DP Fourier series, in the light of the correspondence between FC pairs of DP Fourier series and Taylor series of inner analytic functions, which was established in a previous paper [1]. This discussion was started in that paper, and will be continued here in order to include the more complex and difficult cases. We will assume that the reader is aware of the contents of that paper, and will use without too much explanation the concepts, the definitions and the notations established there. We will limit ourselves here to the explanation of a few abbreviations, such as DP, which stands for Definite Parity, and FC, which stands for Fourier Conjugate, and to the restatement of the more basic concepts.

We refer as the *basic convergence theorem* of complex analysis to the result that, if a complex power series around a point z_0 converges at a point $z_1 \neq z_0$, then it is convergent and absolutely convergent in the interior of a disk centered at $z = z_0$ with its boundary passing through z_1. In addition to this, it converges uniformly on any closed set contained within this open disk. The concept of *inner analytic function* makes reference to a complex function that is analytic at least on the open unit disk, assumes the value zero at $z = 0$, and is the analytic continuation of a real function on the interval $(-1, 1)$ of the real axis. The restriction of this complex function to the unit circle results in a FC pair of DP real functions. The corresponding Taylor series around $z = 0$ converges at least on the open unit disk, assumes the value zero at $z = 0$, and has real coefficients. The restriction of this complex power series to the unit circle results in a FC pair of DP Fourier series.

The concept of *Definite Parity* or DP Fourier series refers to a Fourier series which has only the sine terms, or only the cosine terms, without the constant term, and therefore has a definite parity on the periodic interval $[-\pi, \pi]$. In the same way, DP real functions are those that have definite parity on the periodic interval. The concept of *Fourier Conjugate* or FC trigonometric series refers to a conjugate series that is built from a given DP trigonometric series by the exchange of cosines by sines, or vice-versa. The concept of FC real functions refers to the real functions that FC Fourier series converge to, or more generally to the pair of real functions that generates the common Fourier coefficients of a FC pair of DP trigonometric series, even if the series do not converge.

The concepts of the *degrees of hardness* and of the *degrees of softness* of singularities in the complex plane, which were introduced in the previous paper [1], will be discussed and defined in more precise terms in this paper. The basic concept is that a soft singularity is one where the complex function is still well defined, although not analytic, when one takes the limit to the singular point. On the other hand, a hard singularity is one where the complex function diverges to infinity or does not exist when one takes that limit. A borderline hard singularity is the least hard type of hard singularity, while a borderline soft singularity is the least soft type of soft singularity.

We will consider, then, the convergence of DP Fourier series and of the corresponding complex power series. For organizational reasons this discussion must be separated into several parts. In the previous paper [1] we tackled the extreme cases which we qualify as those of *strong divergence* and *strong convergence*, for which results can be established in full generality. In this paper we will deal with the remaining cases, which we will qualify as those of *weak convergence* and *very weak convergence*. In the case of very weak convergence we will be able to present definite results only for a certain class of DP Fourier series.

2 Weak Convergence

In the previous paper [1] we discussed the issue of the convergence of FC pairs of DP Fourier series such as

$$\begin{aligned} S_{\rm c} &= \sum_{k=1}^{\infty} a_k \cos(k\theta), \\ S_{\rm s} &= \sum_{k=1}^{\infty} a_k \sin(k\theta), \end{aligned}$$

with real Fourier coefficients a_k, for θ in the periodic interval $[-\pi, \pi]$, and of the corresponding complex power series

$$\begin{aligned} S_v &= \sum_{k=1}^{\infty} a_k v^k, \\ S_z &= \sum_{k=1}^{\infty} a_k z^k, \end{aligned}$$

where $v = \exp(\imath\theta)$ and $z = \rho v$, in two extreme cases, those of strong divergence and strong convergence. We established that, in these extreme cases, either the convergence/divergence of S_z or the existence/absence of singularities of the analytic function $w(z)$ that S_z converges to can be used to determine the convergence or divergence of S_v and of the corresponding DP Fourier series over the whole unit circle of the complex plane.

Specifically, we established that the divergence of S_z, or the existence of a singularity of $w(z)$, at any single point strictly within the open unit disk, implies that S_v diverges everywhere on the unit circle, and that the corresponding FC pair of DP Fourier series diverge almost everywhere on the periodic interval. This is what we call strong divergence. To complement this we established that the convergence of S_z at a single point strictly outside the closed unit disk, or the absence of singularities of $w(z)$ on that disk, implies that S_v converges to a C^∞ function everywhere on the unit circle, and that the corresponding FC pair of DP Fourier series converge to C^∞ real functions everywhere on the periodic interval. This is what we call strong convergence.

If the series S_z constructed from an FC pair of DP trigonometric series converges on at least one point on the unit circle, but does not converge at any points strictly outside the closed unit disk, then the situation is significantly more complex. We will see, however, that in essence it can still be completely determined, with some limitations on the set of series involved when establishing the results. We will refer to this situation as the case of *weak convergence*. In this situation the maximum disk of convergence of the complex power series S_z is the open unit disk, and therefore this series converges strongly to an analytic function $w(z)$ strictly within that disk. As is well known, it can be shown that in this case the function $w(z)$ must have at least one singularity on the unit circle.

In terms of the convergence of the series, however, the situation on the unit circle remains undefined. The series S_z may converge or diverge at various points on that circle. The function $w(z)$ cannot be analytic on the whole unit circle, since there must be at least one point on that circle where it has a singularity, in the sense that it is not analytic at that point. This singularity can be of various types, and does not necessarily mean that the function $w(z)$ is not defined at its location. In fact, it is still possible for $w(z)$ to exist everywhere and to be analytic almost everywhere on the unit circle. No general convergence theorem is available for this case, and the analysis must be based on the behavior of the coefficients a_k of the series. It is, therefore, significantly harder to obtain definite results.

While the case of strong divergence is characterized by the fact that the coefficients a_k typically diverge exponentially with k, and the case of strong convergence by the fact that they typically go to zero exponentially with k, in the case of weak convergence the typical behavior is that the coefficients go to zero as a negative power of k. In most of this paper we will in fact limit the discussion to series in which the coefficients go to zero as a negative but not necessarily integer power of k.

2.1 Absolute and Uniform Convergence

The simplest case of what we call here weak convergence is that in which there is absolute and uniform convergence at the unit circle to a function which, although necessarily continuous, is not C^∞. If the coefficients a_k are such that the series S_z is absolutely convergent on a single point z_1 of the unit circle, that is, if the series of absolute values

$$\sum_{k=1}^{\infty} |a_k||z_1|^k = \sum_{k=1}^{\infty} |a_k|,$$

where $|z_1| = 1$, is convergent, then the series S_v is absolutely convergent on the whole unit circle, given that the criterion of convergence is clearly independent of θ in this case. For the same reason, the series in uniformly convergent and thus converges to a continuous function over the whole unit circle. Therefore the corresponding FC Fourier series are also absolutely and uniformly convergent to continuous functions $f_{\rm c}(\theta)$ and $f_{\rm s}(\theta)$ over their whole domain.

In this case, although the function $w(z)$ must have at least one singularity on the unit circle, since the series $S(z)$ converges everywhere that function is still well defined over the whole unit circle, and by Abel's theorem [2] the $\rho \to 1$ limit of the function $w(z)$ from the interior of the unit disk to the unit circle exists at all points of that circle. It follows that the singularity or singularities of $w(z)$ at the unit circle must not involve divergences to infinity. We will classify such singularities, at which the complex function is still well-defined, although non-analytic, as *soft singularities*. Singularities at which the complex function diverges to infinity or cannot be defined at all will be called *hard singularities*.

Since they are convergent everywhere, in this case all DP trigonometric series are DP Fourier series of the continuous function obtained on the unit circle by the $\rho \to 1$ limit of the real and imaginary parts of $w(z)$ from within the unit disk. In short, we have established that weakly-convergent FC pairs of DP Fourier series that converge absolutely and thus uniformly must correspond to inner analytic functions that have only soft singularities on the unit circle. In fact, this would be true of any FC pair of DP Fourier series that simply converges weakly everywhere over the unit circle.

Most of the time this situation can be determined fairly easily in terms of the coefficients a_k of the series. If there is a positive real constant A, a strictly positive number ε and an integer k_m such that for $k > k_m$ the absolute values of the coefficients can be bounded as

$$|a_k| \leq \frac{A}{k^{1+\varepsilon}},$$

then the asymptotic part of the series of the absolute values of the terms of S_v, which is a sum of positive terms and thus increases monotonically, can be bounded from above by a convergent asymptotic integral, as shown in Section A.1 of Appendix A, and thus that series converges. It follows that the S_v series is absolutely and uniformly convergent on the whole unit circle, and therefore so are the two corresponding DP Fourier series.

If the coefficients tend to zero as a power when $k \to \infty$, then it is easy to see that the limiting function defined by the series cannot be C^∞. Since every term-wise derivative of the Fourier series with respect to θ adds a factor of k to the coefficients, a decay to zero as $1/k^{n+1+\varepsilon}$ with $0 < \varepsilon \leq 1$ guarantees that we may take only up to n term-by-term derivatives and still end up with an absolutely and uniformly convergent series. With $n + 1$ differentiations the resulting series might still converge almost everywhere, but that is not certain, and typically at least one of the two DP series in the FC pair will diverge somewhere. With $n + 2$ differentiations the series S_v is sure to diverge everywhere, and the two DP series in the FC pair are sure to diverge almost everywhere, since the coefficients no longer go to zero as $k \to \infty$.

It is easy to see that the points where one of the limiting functions $f_{\rm c}(\theta)$ and $f_{\rm s}(\theta)$ is not differentiable, or those at which there is a singularity in one of its derivatives, must be points where $w(z)$ is singular. At any point of the unit circle where $w(z)$ is analytic it not only is continuous but also infinitely differentiable. We see therefore that the set of points of the unit circle where $f_{\rm c}(\theta)$ and $f_{\rm s}(\theta)$ or their derivatives of any order have singularities of any kind must be exactly the same set of points where $w(z)$ has singularities on that circle. In our present case, since the functions must be continuous, these singularities can only be points of non-differentiability of the functions or points where some of their higher-order derivatives do not exist.

We conclude therefore that the cases in which the convergence is not strong but is still absolute and uniform are easily characterized. The really difficult cases regarding convergence are, therefore, those in which the series are not absolutely or uniformly convergent on the unit circle. These are cases in which $|a_k|$ goes to zero as $1/k$ or slower as $k \to \infty$, as shown in Section A.1 of Appendix A. We will refer to these cases as those of *very weak convergence*. In this case the Fourier series can converge to discontinuous functions, and S_z will typically diverge at some points of the unit circle, at which $w(z)$ has divergent limits and therefore hard singularities. In order to discuss this case we must first establish a few simple preliminary facts, leading to a classification of singularities according to their severity.

3 Classification of Singularities

Let us establish a general classification of the singularities of inner analytic functions on the unit circle. While it is possible that the scheme of classification that we describe here may have its uses in more general settings, for definiteness we consider here only the case of inner analytic functions that have one or more singularities on the unit circle. We also limit our attention to only those singular points, and ignore any singularities that may exist strictly outside the closed unit disk. In order to do this we must first establish some preliminary facts.

To start with, let us show that the operation of logarithmic differentiation stays within the set of inner analytic functions. Let us recall that, if $w(z)$ is an inner analytic function, then it has the properties that it is analytic on the open unit disk, that it is the analytic continuation of a real function on the real interval $(-1, 1)$, and that $w(0) = 0$. These last two are consequences of the fact that its Taylor series has the form

$$w(z) = \sum_{k=1}^{\infty} a_k z^k,$$

with real a_k. We define the logarithmic derivative of $w(z)$ as

$$w^{\cdot}(z) = z \, \frac{dw(z)}{dz},$$

which also establishes our notation for it. We might also write the first logarithmic derivative as $w^{1\cdot}(z)$. Let us show that the logarithmic derivative of an inner analytic function is another inner analytic function. First, since the derivative of an analytic function is analytic in exactly the same domain as that function, and since the identity function z is analytic in the whole complex plane, it follows that if $w(z)$ is analytic on the open unit disk, then so is $w^{\cdot}(z)$. Second, if we calculate $w^{\cdot}(z)$ using the series representation of $w(z)$, which converges in the open unit disk, since $w(z)$ is analytic there, we get

$$w^{\cdot}(z) = z \sum_{k=1}^{\infty} k a_k z^{k-1},$$

since a convergent power series can always be differentiated term-by-term. It follows that, if the coefficients a_k are real, then so are the new coefficients $k a_k$, so that the coefficients of the Taylor series of $w^{\cdot}(z)$ are real, and hence it too is the analytic continuation of a real function on the real interval $(-1, 1)$. Lastly, due to the extra factor of z we have that $w^{\cdot}(0) = 0$. Hence, the logarithmic derivative $w^{\cdot}(z)$ is an inner analytic function.

Next, let us define the concept of logarithmic integration. This is the inverse operation to logarithmic differentiation, with the understanding that we always choose the value zero for the integration constant. The logarithmic primitive of $w(z)$ may be defined as

$$w^{-1\cdot}(z) = \int_0^z dz' \, \frac{1}{z'} \, w(z').$$

Let us now show that the operation of logarithmic integration also stays within the set of inner analytic functions. Note that since $w(0) = 0$ the integrand is in fact analytic on the open unit disk, if we define it at $z = 0$ by continuity, and therefore the path of integration from 0 to z is irrelevant, so long as it is contained within that disk. It is easier to see this using the series representation of $w(z)$, which converges in the open unit disk, since $w(z)$ is analytic there,

$$w^{-1\cdot}(z) = \int_0^z dz' \sum_{k=1}^{\infty} a_k z'^{(k-1)}.$$

It is clear now that the integrand is a power series which converges within the open unit disk, and thus converges to an analytic function there. Since the primitive of an analytic function is analytic in exactly the same domain as that function, it follows that the logarithmic primitive $w^{-1\cdot}(z)$ is analytic on the open unit disk. If we now execute the integration using the series representation, we get

$$w^{-1\cdot}(z) = \sum_{k=1}^{\infty} \frac{a_k}{k} \, z^k,$$

since a convergent power series can always be integrated term-by-term. Since the coefficients a_k are real, so are the new coefficients a_k/k, and therefore $w^{-1\cdot}(z)$ is the analytic continuation of a real function on the real interval $(-1, 1)$. Besides, one can see explicitly that $w^{-1\cdot}(0) = 0$. Therefore, $w^{-1\cdot}(z)$ is an inner analytic function. It is now easy to see also that the logarithmic derivative of this primitive gives us back $w(z)$.

Finally, let us show that the derivative of $w(z)$ with respect to θ is given by the logarithmic derivative of $w(z)$. Since we have that $z = \rho \exp(\imath\theta)$, we may at once write that

$$\begin{aligned} \frac{dw(z)}{d\theta} &= \frac{dz}{d\theta} \frac{dw(z)}{dz} \\ &= \imath z \frac{dw(z)}{dz} \\ &= \imath w^{\cdot}(z). \end{aligned}$$

In the limit $\rho \to 1$, where and when that limit exists, the derivative of $w(z)$ with respect to θ becomes the derivatives of the limiting functions $f_{\rm c}(\theta)$ and $f_{\rm s}(\theta)$ of the FC pair of DP Fourier series. The factor of $\imath$ effects the interchange of real and imaginary parts, and the change of sign, that are consequences of the differentiation of the trigonometric functions.

We see therefore that, given an inner-analytic function and its set of singularities on the unit circle, as well as the corresponding FC pair of DP Fourier series, we may at once define a whole infinite chain of inner-analytic functions and corresponding DP Fourier series, running by differentiation to one side and by integration to the other, indefinitely in both directions. We will name this an *integral-differential chain*. We are now ready to give the complete formal definition of the proposed classification of the singularities of inner analytic functions $w(z)$ on the unit circle. Let z_1 be a point on the unit circle. We start with the very basic classification which was already mentioned.

- A singularity of $w(z)$ at z_1 is a *soft singularity* if the limit of $w(z)$ from within the unit disk to that point exists and is finite.

- A singularity of $w(z)$ at z_1 is a *hard singularity* if the limit of $w(z)$ from within the unit disk to that point does not exist, or is infinite.

Next we establish a gradation of the concepts of hardness and softness of the singularities of $w(z)$. To each singular point z_1 we attach an integer giving either its *degree of hardness* or its *degree of softness*. In order to do this the following definitions are adopted.

- A single logarithmic integration of $w(z)$, that does not change the hard/soft character of a singularity, increases the degree of softness of that singularity by one, if it is soft, or decreases the degree of hardness of that singularity by one, if it is hard.

- A single logarithmic differentiation of $w(z)$, that does not change the hard/soft character of a singularity, increases the degree of hardness of that singularity by one, if it is hard, or decreases the degree of softness of that singularity by one, if it is soft.

- Given a soft singularity at z_1, if $w(z)$ can be logarithmically differentiated indefinitely without that singularity ever becoming hard, then we say that it is an *infinitely soft singularity*. This means that its degree of softness is infinite.

- Given a hard singularity at z_1, if $w(z)$ can be logarithmically integrated indefinitely without that singularity ever becoming soft, then we say that it is an *infinitely hard singularity*. This means that its degree of hardness is infinite.

- A singularity of $w(z)$ at z_1 is a *borderline soft* one if it is a soft singularity and a single logarithmic differentiation of $w(z)$ results in a hard singularity at that point. A borderline soft singularity has degree of softness zero.

- A singularity of $w(z)$ at z_1 is a *borderline hard* one if it is a hard singularity and a single logarithmic integration of $w(z)$ results in a soft singularity at that point. A borderline hard singularity has degree of hardness zero.

Finally, the following rules are adopted regarding the superposition of several singularities as the same point, brought about by the addition of functions.

- If there is more than one hard singularity of $w(z)$ superposed at z_1, then the result is a hard singularity and its degree of hardness is that of the component singularity with the largest degree of hardness.

- If there is more than one soft singularity of $w(z)$ superposed at z_1, then the result is a soft singularity and its degree of softness is that of the component singularity with the smallest degree of softness.

- The superposition of hard singularities and soft singularities of $w(z)$ at z_1 results in a hard singularity at z_1, with the degree of hardness of the component singularity with the largest degree of hardness.

It is not difficult to see that this classification spans all existing possibilities in so far as the possible types of singularity go. First, given a point of singularity, either the limit of the function to that point from within the open unit disk exists or it does not. There is no third alternative, and therefore every singularity is either soft or hard. Second, given a soft singularity, either it becomes hard after a certain finite number of logarithmic differentiations of $w(z)$, or it does not. Similarly, given a hard singularity, either it becomes soft after a certain finite number of logarithmic integrations of $w(z)$, or it does not. In either case there is no third alternative. If the soft or hard character never changes, then we classify the singularity as infinitely soft or infinitely hard, as the case may be. Otherwise, we assign to it a degree of softness or hardness by counting the number n of logarithmic differentiations or logarithmic integrations required to effect its change of character, and assigning to it the number $n-1$ as the degree of softness or hardness, as the case may be.

We now recall that there is a set of hard singularities which is already classified, by means of the concept of the Laurent expansion around an isolated singular point. If a singularity is isolated in two ways, first in the sense that there is an open neighborhood around it that contains no other singularities, and second that it is possible to integrate along a closed curve around it which is closed in the sense that it does not pass to another leaf of a Riemann surface when it goes around the point, then one may write a convergent Laurent expansion for the function around that point. This leads to the concepts of poles of finite orders and of essential singularities. In particular, it implies that any analytic function that has a pole of finite order at the point z_1 can be written around that point as the sum of a function which is analytic at that point and a finite linear combination of the singularities

$$\frac{1}{(z-z_1)^n},$$

for $1 \leq n \leq n_{\rm o}$, where $n_{\rm o}$ is the order of the pole. We can use this set of singularities to illustrate our classification. For example, if we have an inner analytic function $w(z)$ with a simple pole $1/(z-z_1)$ for z_1 on the unit circle, which is a hard singularity with degree of hardness 1, then the logarithmic derivative of $w(z)$ has a double pole $1/(z-z_1)^2$ at that point, an even harder singularity, of degree of hardness 2. Further logarithmic differentiations of $w(z)$ produce progressively harder singularities $1/(z-z_1)^n$, where n is the degree of hardness of the singularity. We see therefore that multiple poles fit easily and comfortably into the classification scheme. We may now proceed to examine this chain of singularities in the other direction, using logarithmic integration in order to do this.

The logarithmic primitive of the function $w(z)$ mentioned above has a logarithmic singularity $\ln(z-z_1)$ at that point, which is the weakest type of hard singularity in this type of integral-differential chain. Another logarithmic integration produces a soft singularity such as $(z-z_1)\ln(z-z_1)$, which displays no divergence to infinity. This establishes therefore that $\ln(z-z_1)$ is a borderline hard singularity, with degree of hardness 0. If we now proceed to logarithmically differentiate the resulting function, we get back the hard singularity $\ln(z-z_1)$. This establishes therefore that $(z-z_1)\ln(z-z_1)$ is a borderline soft singularity, with degree of softness 0. This illustrates the transitions between hard and soft singularities, and also justifies our attribution of degrees of hardness to the multiple poles, as we did above. Further logarithmic integrations produce progressively softer singularities such as $(z-z_1)^2\ln(z-z_1)$, and so on, were we consider only the hardest or least soft singularity resulting from each operation and ignore regular terms, leading to the general expression

$$(z-z_1)^{n+1}\ln(z-z_1),$$

where n is the degree of softness. This completes the examination of the singularities of this particular type of integral-differential chain. Note that these soft singularities are isolated in the sense that there is an open neighborhood around each one of them that contains no other singularities, but not in the sense that one can integrate in closed curves around them. This is so because the domains of these functions are in fact Riemann surfaces with infinitely many leaves, and a curve which is closed in the complex plane is not really closed in the domain of the function.

Although this chain of singularities exhausts the possibilities so far as one is limited to integral-differential chains containing isolated hard singularities of single-valued functions, there are many other possible chains of singularities, if one starts with hard singularities having non-trivial Riemann surfaces, for example such as

$$\frac{\ln(z - z_1)}{(z - z_1)^n},$$

for $n \geq 1$. One might consider also the more general form

$$(z - z_1)^n \ln^m(z - z_1)$$

for the singularities, where $m \geq 0$ and n in any integer, positive or negative. This generates quite a large set of possible types of singularity, both soft and hard.

To complete the picture in our exemplification, a simple and widely known example of an infinitely hard singularity is an essential singularity such as $\exp[1/(z-z_1)]$. On the other hand, an infinitely soft singularity is not such a familiar object. One interesting example will be discussed in Section A.2 of Appendix A.

It is important to note that almost all convergent DP Fourier series will be related to inner analytic functions either with only soft singularities on the unit circle or with at most borderline hard singularities, which will therefore all have non-trivial Riemann surfaces as their domains. Since in our analysis here we are bound within the unit disk, and will at most consider limits to the unit circle from within that disk, this is not of much concern to us, because in this case we never go around one of these singularities in order to change from one leaf of the Riemann surface to another. The value of the function $w(z)$ within the unit disk is defined by its value a the origin, and this determines the leaf of each Riemann surface which is to be used within the disk. We must always consider that the branching lines of all such branching points at the unit circle extend outward from the unit circle, towards infinity.

4 Convergence and Singularities

Let us examine the effects on the corresponding series and functions of each one of these two related types of differentiation and integration operations in turn. First the effect of differentiations and integrations with respect to θ acting on the DP Fourier series, and then the effect of logarithmic differentiations and integrations acting on the corresponding inner analytic functions $w(z)$.

On the Fourier side of this discussion, it is clear from the structure of the DP Fourier series that each differentiation with respect to θ adds a factor of k to the a_k coefficients, and thus makes them go to zero slower, or not at all, as $k \to \infty$. This either reduces the rate of convergence of the series or makes them outright divergent. Integration with respect to θ, on the other hand, has the opposite effect, since it adds to the a_k coefficients a factor of $1/k$, and thus makes them go to zero faster as $k \to \infty$. This always increases the rate of convergence of the series. On the inner-analytic side of the discussion, each logarithmic differentiation increases the degree of hardness of each singularity, while logarithmic integration decreases it. At the same time, these operations work in the opposite way on the degrees of softness. No point of singularity over the unit circle ever vanishes or appears as a result of these operations. Only the degrees of hardness and softness change.

We are now in a position to use these preliminary facts to analyze the question of the convergence of the series on the unit circle. While the derivations and integrations with respect to θ change the convergence status of the DP Fourier series on the unit circle, the related logarithmic derivations and integrations of the corresponding inner analytic function do not change the analyticity or the set of singular points of that function. The only thing that these logarithmic operations do change is the *type* of the singularities over the unit circle, as described by their degrees of hardness or softness. Hence, there must be a relation between the mode of convergence or lack of convergence of the DP Fourier series on the unit circle and the nature of the singularities of the inner analytic function on that circle. It is quite clear that the rate of convergence and the very existence of convergence of the DP Fourier series are tied up to the degree of hardness or softness of the singularities present. The less soft the singularities, the slower the convergence, leading eventually to hard singularities and to the total loss of convergence.

Note that the relation between the singularities of $w(z)$ on the unit circle and the convergence of the DP Fourier series is non-local, because processes of differentiation or integration will change the degree of hardness or softness only locally at the singular points, but will affect the speed of convergence to zero of the coefficients a_k, which has its effects on the rate of convergence of the DP Fourier series everywhere over the unit circle. Since the relation between the hardness or softness of the singularities and the convergence of the DP Fourier series is non-local, the rate of convergence or the lack of it will be ruled by the hardest or least soft singularity or set of singularities found anywhere over the whole unit circle. We will call these the *dominant singularities*. Therefore, from now on we will think in terms of the dominant singularity or set of singularities which exists on that circle.

The problem of establishing a general, complete and exact set of criteria determining the convergence or arbitrary DP Fourier series with basis on the set of dominant singularities of the corresponding inner analytic function is, so far as we can tell, an open one. We will, however, be able to classify and obtain the convergence criteria for a fairly large class of DP Fourier series. In order to establish this class of series, consider the set of all possible integral-differential chains involving the series S_z, S_v, $S_{\rm c}$ and $S_{\rm s}$, and the respective functions $w(z)$, $f_{\rm c}(\theta)$ and $f_{\rm s}(\theta)$. Let us use S_v to characterize the elements of these chains. Since the operations of logarithmic differentiation and logarithmic integration always produce definite and unique results, it is clear the each series S_v belongs to only one of these chains. Let us now select from the set of all possible integral-differential chains those that satisfy the following two conditions.

- There is in the chain a series S_v with coefficients a_k that can be bounded in the following way: there exist positive real constants $A_{(-)}$ and $A_{(+)}$, a minimum value k_m of k and a positive real number $0 < \varepsilon < 1$ such that for $k > k_m$

$$\frac{A_{(-)}}{k} \leq |a_k| \leq \frac{A_{(+)}}{k^{\varepsilon}}.$$

- The series S_v qualified in the previous condition diverges to infinity on at least one point of the unit circle.

We will call the integral-differential chains that satisfy these conditions *regular integral-differential chains*. We will adopt as a shorthand for the first condition the statement that $|a_k|$ behaves as $1/k^p$ for large values of k, or $|a_k| \propto 1/k^p$, where $0 < p \leq 1$. What the condition means is that, while the coefficients go to zero as $k \to \infty$, they do it sufficiently slowly to prevent the series S_v from being absolutely and uniformly convergent. Therefore the series S_v may still converge, but does not converge absolutely. Because the series satisfies the second condition we know, from the extended version of Abel's theorem [2], that the corresponding inner analytic function has a divergent limit going to infinity, on at least one point of the unit circle. Therefore, it has at least one hard singularity on that circle. We see then that the set of dominant singularities that the inner analytic function has on the unit circle is necessarily a set of hard singularities. We will denote the series that satisfies these conditions in any given regular integral-differential chain by $S_{v,h0}$, and the corresponding inner analytic function by $w_{h0}(z)$.

The next series in the chain, obtained from this one by logarithmic integration, which is the same as integration with respect to θ, has coefficients that behave as $|a_k| \propto 1/k^p$ with $1 < p \leq 2$, and is therefore absolutely and uniformly convergent everywhere. It follows therefore that all the singularities of the corresponding inner analytic function are soft. We will denote the series obtained from $S_{v,h0}$ in this way by $S_{v,s0}$, and the corresponding inner analytic function by $w_{s0}(z)$. The set of dominant singularities of $w_{s0}(z)$ is thus seen to be a set of soft singularities. It follows that the dominant singularities of $w_{h0}(z)$, which are hard and became soft by means of a single operation of logarithmic integration, constitute a set of borderline hard singularities. Since we may go back from $w_{s0}(z)$ to $w_{h0}(z)$ by a single operation of logarithmic differentiation, it also follows that the set of dominant singularities of $w_{s0}(z)$ is a set of borderline soft singularities.

If we go further along in either direction of the chain, in the integration direction all subsequent series $S_{v,sn}$ are also absolutely and uniformly convergent, to continuous functions that are everywhere C^n and almost everywhere C^{n+1}, typically sectionally C^{n+1}, where n is the degree of softness. The corresponding inner analytic functions $w_{sn}(z)$ have only soft singularities on the unit circle. In the other direction, the next series in the chain, denoted by $S_{v,h1}$, obtained from $S_{v,h0}$ by logarithmic differentiation, has coefficients that behave as $|a_k| \propto k^p$ with $0 \leq p < 1$, and is therefore divergent everywhere, since the coefficients do not go to zero as $k \to \infty$. The same can be said of all the subsequent elements $S_{v,hn}$ of the chain in this direction. The corresponding inner analytic functions $w_{hn}(z)$ have sets of dominant singularities consisting of hard singularities. We arrive therefore at the following scheme of convergence diagnostics, for series S_v on regular integral-differential chains, based on the behavior of the dominant singularities of $w(z)$.

- If the dominant singularities of $w(z)$ are hard with degree of hardness $n \geq 1$, then the series S_v is everywhere divergent.
- If the dominant singularities of $w(z)$ are borderline hard ones, then S_v *may* still be convergent almost everywhere to a discontinuous function, but will diverge in one or more points. If the number of dominant singularities is finite, then the limiting function will be sectionally continuous and differentiable.
- If the dominant singularities of $w(z)$ are borderline soft ones, then S_v is everywhere convergent to an everywhere continuous but not everywhere differentiable function. If the number of dominant singularities is finite, then the limiting function will be sectionally differentiable.
- If the dominant singularities of $w(z)$ are soft with degree of softness $n \geq 1$, then the series S_v is everywhere convergent to a C^n function, which is also C^{n+1} almost everywhere. If the number of dominant singularities is finite, then the limiting function will be sectionally C^{n+1}.

Given the convergence status of S_v, corresponding conclusions can then be drawn for the FC pair of DP Fourier series $S_{\rm c}$ and $S_{\rm s}$. Observe that in this argument the facts about the convergence of the DP Fourier series are in fact feeding back into the question of the convergence of the power series S_z at the rim of its maximum disk of convergence. It follows therefore that this classification can be understood as a set of statements purely in complex analysis, since it also states conditions for the convergence or divergence of the Taylor series S_z over the unit circle, in the cases when that circle is the boundary of its maximum disk of convergence.

Note that in the case in which the dominant singularities are borderline hard ones there is as yet no certainty of convergence almost everywhere. Therefore in this respect this alternative must be left open here, and will be discussed in the next section for some classes of series. It is an interesting question whether or not there are series S_v with coefficients that behave as $|a_k| \propto 1/k^p$ with $0 < p \leq 1$ and that do not diverge to infinity anywhere. The common examples all seem to diverge as we have assumed here. It seems to be difficult to find an example that has the opposite behavior, but we can offer no proof one way of the other. Therefore, we must now leave this here as a mere speculation.

Observe that, since the inner analytic function can be integrated indefinitely, as many times as necessary, we have here another way, at least in principle, to get some information about the function that originated an arbitrarily given DP Fourier series, besides taking limits of $w(z)$ from within the open unit disk. We may construct the series S_z from the coefficients and, if the corresponding inner analytic function $w(z)$ is in fact analytic on the open unit disk, with some upper bound for the hardness of the singularities on the unit circle, then we may logarithmically integrate it as many times as necessary to reduce all the singularities over the unit circle to soft singularities. We will then have corresponding series S_c and S_s that are absolutely and uniformly convergent over the unit circle, with continuous functions as their limits.

The original function is then a multiple derivative with respect to θ of one of these continuous functions. Typically we will not be able to actually take these derivatives at the unit circle, but we will always be able to take the corresponding derivatives of $w(z)$ within the open unit disk, arbitrarily close to the unit circle, so that at least we will have a chance to understand the origin of the problem, which might give us an insight into the structure of the application, and related circumstances, that generated the given DP Fourier series. This procedure will fail only if the inner analytic function turns out to have an infinite number of singularities on the unit circle, with degrees of hardness without an upper bound, or an infinitely hard singularity, such as an essential singularity, anywhere on the unit circle.

5 Monotonic Series

The main remaining problem in our classification of convergence modes in terms of the dominant singularities is to show that if those singularities are borderline hard ones then the S_v series is still convergent almost everywhere. While we are not able to prove this in general, even within the realm of the regular integral-differential chains, there are some rather large classes of DP Fourier series for which it is possible to prove the convergence almost everywhere. These series satisfy all the conditions imposed in the previous section, which means that they belong to regular integral-differential chains. We will also be able to establish the existence, location and character of the dominant singularities of the corresponding inner analytic functions $w(z)$.

We will call these series *monotonic series*, which refers to the fact that they have coefficients a_k that behave monotonically with k. Other series can also be built from these monotonic series by means of finite linear combinations, which will share their properties regarding convergence and dominant singularities, but which are not themselves monotonic. We will call these *extended monotonic series*. The method we will use to establish proof of convergence will lead to the concept of *singularity factorization*, which we will later generalize. This is a method for evaluating the series which is algorithmically useful, and can be used safely in very general circumstances.

5.1 A Monotonicity Test

Within the set of all series S_v which are very weakly convergent, there is a subset of series that all converge almost everywhere, as we will now show. This is the class of series S_v that have coefficients a_k that converge monotonically to zero. This monotonicity of the coefficients can, therefore, be used as a convergence test. No statement at all has to be made about the speed of convergence to zero, but we may as well focus our attention on series of type $S_{v,h0}$, with coefficients behaving as $|a_k| \propto 1/k^p$ with $0 < p \leq 1$, which are those for which this analysis is most useful, since in this case the sum

$$\sum_{k=1}^{\infty} |a_k|$$

diverges to infinity and the series is therefore not absolutely or uniformly convergent. For simplicity we will take the case in which $a_k > 0$ for all k, but the argument can be easily generalized to several other cases, as will be discussed later on. Let us consider then the series

$$S_v = \sum_{k=1}^{\infty} a_k v^k,$$

where $a_k > 0$ for all k, $a_{k+1} \leq a_k$ for all k and

$$\lim_{k \to \infty} a_k = 0.$$

Such a series can be shown to converge for $\theta \neq 0$ by the use of the Dirichlet test. For $\theta = 0$ it diverges to positive infinity along the real axis, and therefore satisfies the hypotheses defining a series of type $S_{v,h0}$ in a regular integral-differential chain. It is not too difficult to show that, for any non-zero value of θ in the periodic interval, the Dirichlet partial sums

$$D_N = \sum_{k=1}^{N} v^k,$$

are contained within a closed disk of radius

$$R = \frac{1}{2|\sin(\theta/2)|},$$

centered at the point

$$C = \frac{1}{2} + \imath \frac{1}{2\tan(\theta/2)}$$

of the complex plane, for all values of N. The absolute values $|D_N|$ of the Dirichlet partial sums are therefore bounded by a constant for all N, and since the coefficients go monotonically to zero, the Dirichlet test applies and the series S_v is convergent.

We will, however, demonstrate the convergence in another way, which we believe to be more fruitful, and more directly related to our problem here. What we will do is to construct another expression, involving another DP Fourier series, that converges to the same function and, unlike S_v, does so absolutely and uniformly almost everywhere. Consider then the following algebraic passage-work for our series, for $\theta \neq 0$, which is equivalent to $v \neq 1$. We start by multiplying and dividing the series by the factor $(v-1)$, and distributing the one in the numerator,

$$\begin{aligned} S_v &= \frac{v-1}{v-1} \sum_{k=1}^{\infty} a_k v^k \\ &= \frac{1}{v-1} \left[\sum_{k=1}^{\infty} a_k v^{k+1} - \sum_{k=1}^{\infty} a_k v^k \right]. \end{aligned}$$

We now change the index of the first series, in order to be able to join the two resulting series,

$$\begin{aligned} S_v &= \frac{1}{v-1} \left[\sum_{k=2}^{\infty} a_{k-1} v^k - \sum_{k=1}^{\infty} a_k v^k \right] \\ &= \frac{1}{v-1} \left[-a_1 v + \sum_{k=2}^{\infty} (a_{k-1} - a_k) v^k \right]. \end{aligned}$$

If we now define the new coefficients $b_1 = -a_1$ and $b_k = a_{k-1} - a_k$ for $k > 1$, we have a new series C_v, which we name the *center series* of S_v, so that we have

$$\begin{aligned} S_v &= \frac{1}{v-1} C_v, \\ C_v &= \sum_{k=1}^{\infty} b_k v^k. \end{aligned}$$

The name we chose for this series comes from the fact that it describes the relatively small drift in the complex plane of the instantaneous center of rotation of the convergence process of the S_v series. Specially for $a_k \propto 1/k^p$ with the values of p closer to zero, the convergence process of the S_v series proceeds in a long, slow spiral around the limiting point, while that of the C_v series goes more or less directly to it. Let us now show that C_v is absolutely and thus uniformly convergent. We simply consider the series $\overline{C}_v$ of the absolute values of the terms of C_v. Since $|v| = 1$, we get

$$\begin{aligned} \overline{C}_v &= \sum_{k=1}^{\infty} |b_k| \\ &= a_1 + \sum_{k=2}^{\infty} |a_{k-1} - a_k|. \end{aligned}$$

Since the coefficients a_k decrease monotonically to zero, we have $|a_{k-1} - a_k| = a_{k-1} - a_k$ and thus we may write

$$\begin{aligned} \overline{C}_v &= a_1 + \sum_{k=2}^{\infty} a_{k-1} - \sum_{k=2}^{\infty} a_k \\ &= a_1 + \sum_{k=1}^{\infty} a_k - \sum_{k=2}^{\infty} a_k \\ &= 2a_1. \end{aligned}$$

Since a_1 is some finite real number, this establishes an upper bound for $\overline{C}_v$. Since this series is a sum of positive terms and thus monotonically increasing, this implies that it converges and therefore that C_v is absolutely convergent. Since the bound and therefore the criterion of convergence are independent of θ, the convergence is also uniform. Therefore, for $v \neq 1$ we may evaluate S_v by first evaluating C_v and then multiplying the result by the simple pole $1/(v-1)$.

This proves not only the absolute and uniform convergence of the series C_v everywhere on the unit circle, it also implies the convergence of the original series S_v at all points except the special point $v = 1$, which corresponds to $\theta = 0$. Obviously the S_v series does not converge absolutely or uniformly, but it does converge at all points except $v = 1$. We see therefore that the monotonicity of the coefficients can be used as a test for simple point-wise convergence of the S_v series almost everywhere on the unit circle. The special point, were the S_v series diverges, is easily identified as the point where $w(z)$ has a borderline hard singularity, since a single logarithmic integration of $w(z)$ will necessarily result in a soft singularity there.

If we extend the series C_v to a full complex power series C_z on the unit disk, just as we did before in the case of S_v and S_z, we immediately see that

$$C_z = (z-1)S_z.$$

Note that C_z satisfies all the necessary conditions for convergence to an inner analytic function. Since S_z converges to an analytic function on the open unit disk and $(z-1)$ is analytic on the whole complex plane, it follows that C_z also converges to an analytic function on that disk. In addition to this, since $(z-1)$ reduces to a real function on the real line and S_z reduces to a real function on the real interval $(-1, 1)$, so does C_z. Finally, it is clear that since S_z is zero at $z = 0$, so is C_z. Therefore C_z converges to an inner analytic function $\gamma(z)$, so that we have

$$\gamma(z) = (z-1)w(z).$$

Since C_z is absolutely and uniformly convergent on the unit circle, the function $\gamma(z)$ can have only soft singularities on that circle. However, S_z diverges to infinity at one point on the unit circle and, due to the extended version of Abel's theorem [2], it follows that $w(z)$ must have a hard singularity at that point. We see therefore that in this case the multiplication by $(z-1)$ has the same effect of a logarithmic integration. While the function $w(z)$ has a single borderline hard singularity at $z = 1$, the function $\gamma(z)$ has a borderline soft singularity at that point.

We may conclude here that this whole class of series satisfies our hypotheses defining a series of type $S_{v,h0}$ in a regular integral-differential chain, as well as that all of them converge almost everywhere. Besides, all the series in this whole class have a single dominant borderline hard singularity located at $z = 1$. Note that even if a monotonic series has coefficients that go to zero as a power in ways other than $1/k^p$ with $0 < p \leq 1$, the construction of C_z still applies. In this case $w(z)$ will still have a dominant singularity at $z = 1$, although no longer a borderline hard one, but a soft one instead.

5.2 Monotonicity Extensions

The argument given in the last subsection can be generalized in several ways, and the generalizations constitute proof of the convergence of wider classes of series of type $S_{v,h0}$. For example, it can be trivially generalized to series with negative a_k for all k, converging to zero from below. It is also trivial that it can be generalized to series which include a non-monotonic initial part, up to some minimum value k_m of k. In a less trivial, but still simple way, the argument can be generalized to series with only odd-k terms, such as the square wave, or with only even-k terms, such as the two-cycle sawtooth wave. We can prove this in a simple way by reducing these cases to the previous one. If we have a complex power series given by

$$S_z = \sum_{j=1}^{\infty} a_k z^k,$$

where $k = 2j$, we may simply define new coefficients $a'_j = a_k$ and a new complex variable $z' = z^2$, in terms of which the series can now be written as

$$S_z = \sum_{j=1}^{\infty} a'_j z'^j,$$

thus reducing it to the previous form, in terms of the variable z'. So long as the non-zero coefficients a_k tend monotonically to zero as $k \to \infty$, we have that the coefficients a'_j tend monotonically to zero as $j \to \infty$, and the previous result applies. The same is true if we have a complex power series given by

$$S_z = \sum_{j=0}^{\infty} a_k z^k,$$

where $k = 2j + 1$, since we may still define new coefficients $a'_j = a_k$ and the new complex variable $z' = z^2$, in terms of which the series can now be written as

$$S_z = z \sum_{j=0}^{\infty} a'_j z'^j,$$

so that once more, so long as the non-zero coefficients tend monotonically to zero, the previous result applies. We say that series such as these have non-zero coefficients with a constant step 2. The only important difference that comes up here is that the special points on the unit circle are now defined by $z' = 1$, which means that $z^2 = 1$ and hence that $z = \pm 1$. Therefore, in these cases one gets two special points on the unit circle, instead of one, namely $\theta = 0$ and $\theta = \pm\pi$, at which we have dominant singularities, which will be borderline hard singularities so long as $a_j \propto 1/j^p$ with $0 < p \leq 1$.

In addition to this, series with step 1 and coefficients that have alternating signs, such as $a_k = (-1)^k b_k$ with monotonic b_k, which are therefore not monotonic series, can be separated into two sub-series with step 2, one with odd k and the other with even k, and since the coefficients of these two sub-series are monotonic, then the result holds for each one of the two series, and hence for their sum. In this case we will have two dominant singularities in each one of the components series, located at $z = \pm 1$. However, sometimes the two singularities at $z = 1$, one in each component series, may cancel off and the original series may have a single dominant singularity located at $z = -1$. One can see this by means of a simple transformation of variables,

$$(-1)^k b_k z^k = b_k z'^k,$$

where $z' = -z$, so that $z' = 1$ implies $z = -1$. Series with step 2 and coefficients that have alternating signs, such as $a_k = (-1)^j b_k$ with $k = 2j$ or $k = 2j + 1$ and monotonic b_k, which are also not monotonic themselves, can be separated into two sub-series with step 4, and since the coefficients of these two sub-series are monotonic, then the result holds for each one of the two series, and hence for their sum. In this case we will have four dominant singularities in each one of the components series, located at $z = \pm 1$ and $z = \pm \imath$. However, sometimes the singularities at $z = \pm 1$ of the two component series may cancel off and the original series may have only two dominant singularity located at $z = \pm \imath$. One can see this by means of another simple transformation of variables,

$$(-1)^j b_k z^{2j} = b_k z'^j,$$

where $z' = -z^2$, so that $z' = 1$ implies $z^2 = -1$ and hence $z = \pm \imath$. In fact, the result can be generalized to series with non-zero terms only at some arbitrary regular interval Δk, that is, having non-zero terms with some constant step other than 2. If we have a complex power series given by

$$S_z = \sum_{j=0}^{\infty} a_k z^k,$$

where $k = k_0 + pj$, for some strictly positive integer k_0 and where the step p is another strictly positive integer, we may simply define new coefficients $a'_j = a_k$ and a new complex variable $z' = z^p$, in terms of which the series can now be written as

$$S_z = z^{k_0} \sum_{j=0}^{\infty} a'_j z'^j,$$

thus reducing it to the previous form, in terms of the variable z'. So long as the non-zero coefficients a_k tend monotonically to zero as $k \to \infty$, we have that the coefficients a'_j tend monotonically to zero as $j \to \infty$, and the previous result applies. In this case the special points over the unit circle are given by $z^p = 1$, and there are, therefore, p such points, including $z = 1$, uniformly distributed along the circle. If combined with alternating signs, these series have special points given by $z^p = -1$, and once again there are p such points uniformly distributed along the circle. Note that the number of dominant singularities on the unit circle increases with the step p, and that they are homogeneously distributed along that circle.

Finally, one may consider building finite superpositions of the series in all the previous cases discussed so far. Since each component series converges almost everywhere and has a finite number of dominant singularities, these superpositions of series will all converge, and will all have a finite number of dominant singularities on the unit disk. Since all the component series are of type $S_{v,h0}$, and hence have dominant singularities which are borderline hard ones, the dominant singularities of the superpositions will always be at most borderline hard ones. If the dominant singularities are in fact borderline hard ones, then we will call these series *extended monotonic series*. Each one of these series generates a different regular integral-differential chain, and this defines a rather large set of series that can be classified according to our scheme, relating their mode of convergence and the dominant singularities on the unit circle.

6 Factorization of Singularities

One of the interesting facts that follow from the analysis in the previous paper [1] is that, since the limiting function of a DP Fourier series is always given by the limit of the corresponding inner analytic function from within the open unit disk to the unit circle, in any open subset of that circle where $w(z)$ is analytic it is also C^∞ along θ. Therefore a DP Fourier series that converges in a piecewise fashion between two consecutive singularities of $w(z)$ does so to a piecewise section of a C^∞ function. This means that it should be possible to recover the C^∞ function involved in each section, and also that it should be possible to represent them by series that converge at a faster rate and can thus be differentiated at least a few times. In this section we will show how one can accomplish the latter goal.

We start with a simple case, which in fact we have already demonstrated completely in the previous section. The proof of convergence of DP Fourier series with monotonic coefficients described in Subsection 5.1 can be understood as a process of factorization of the singularity of the inner analytic function $w(z)$. Interpreted in terms of S_z we may write the relation between that series and the corresponding center series C_z as

$$\begin{aligned} S_z &= \frac{1}{z-1}\,C_z, \\ C_z &= \sum_{k=1}^{\infty} b_k z^k. \end{aligned}$$

As we have shown before, since S_z converges to an inner analytic function $w(z)$, so does C_z, and hence we have

$$w(z) = \frac{1}{z-1}\,\gamma(z),$$

where C_z converges to $\gamma(z)$. What was done here is to factor out of S_z a simple pole at the point $z = 1$. Hence the original series S_v, which is not absolutely or uniformly convergent and is associated to an inner analytic function that has a borderline hard singularity at $z = 1$, is translated into a series C_v which is absolutely and uniformly convergent, and that is associated to an inner analytic function that has a borderline soft singularity at that point. Note that the $S_z \to C_z$ transformation does not change the maximum disk of convergence or the location of any singularities. Just like logarithmic integration, it just softens the existing singularities.

6.1 General Singularity Factorization

If we think about our general scheme of classification of singularities and modes of convergence, we can see that so long as that scheme holds this process of factoring out singularities should always work, regardless of any hypothesis about the coefficients, such as that they be monotonic. So long as the coefficients of the original DP trigonometric series lead to the construction of an inner analytic function, and so long as that inner analytic function has at most a finite set of dominant singularities over the unit circle, which are not infinitely hard ones such as essential singularities, it should be possible to do this, and hence produce another related series in which the dominant singularities are softened.

Here is how this procedure works. Given a certain DP Fourier series with coefficients a_k, we construct the series S_z and thus the inner analytic function $w(z)$ and determine the set of dominant singularities that it has on the unit circle, which we assume are N in number. Independently of the degree of hardness or softness of these singularities, we now introduce simple poles at each dominant singularity,

$$\begin{aligned} S_z &= \frac{(z-z_1)\ldots(z-z_N)}{(z-z_1)\ldots(z-z_N)}\,S_z \\ &= \frac{1}{(z-z_1)\ldots(z-z_N)}\,C_z, \end{aligned}$$

where the new series is defined by

$$C_z = P_N(z) S_z,$$

where $P_N(z)$ is a polynomial of the order indicated,

$$P_N(z) = (z - z_1) \ldots (z - z_N).$$

This polynomial and the original series S_z can then be manipulated algebraically in order to produce an explicit expression for the new series C_z, which we will still call the center series of S_z. Let us show that C_z converges to an inner analytic function $\gamma(z)$. Since S_z converges to an analytic function $w(z)$ on the open unit disk, it is at once apparent that C_z also converges to an analytic function on that disk, since $P_N(z)$ is a polynomial and hence an analytic function over the whole complex plane. Also, since $S_z = 0$ at the point $z = 0$, it follows at once that $C_z = 0$ on that same point.

Let us now recall that S_z is a power series generated by a FC pair of DP Fourier series, and therefore that its real and imaginary parts have definite parities. Therefore, the inner analytic function $w(z)$ that it converges to also has real and imaginary parts with definite parities. As we showed before, its real part is even on θ, and its imaginary part is odd on θ. Therefore, the singularities of the function $w(z)$ must come in pairs, unless they are located at $\theta = 0$ or $\theta = \pm\pi$. This means that, if there is a singularity at a point z_1 on the unit circle away from the real axis, then there is an essentially identical one at z_1^*,

$$\begin{aligned} z_1 &= \cos(\theta_1) + \imath \sin(\theta_1) \quad \Rightarrow \\ z_1^* &= \cos(\theta_1) - \imath \sin(\theta_1) \\ &= \cos(-\theta_1) + \imath \sin(-\theta_1), \end{aligned}$$

possibly with the overall sign reversed. It follows that, if we want to factor out the singularities on both points, we must choose the factors that constitute $P_N(z)$ in pairs of factors at complex-conjugate points, except possibly for a couple of points over the real axis. Assuming for example that z_2 is real, we have to use something like

$$P_N(z) = (z - z_1)(z - z_1^*) \times (z - z_2) \times (z - z_3)(z - z_3^*) \times \ldots .$$

If we restrict the polynomial to the real axis we get

$$P_N(x) = (x - z_1)(x - z_1^*) \times (x - z_2) \times (x - z_3)(x - z_3^*) \times \ldots .$$

If we now take the complex conjugate of $P_N(x)$ we see that in fact nothing changes,

$$P_N^*(x) = (x - z_1^*)(x - z_1) \times (x - z_2) \times (x - z_3^*)(x - z_3) \times \ldots .$$

Since we thus conclude that $P_N^*(x) = P_N(x)$, it follows that $P_N(x)$ is a real polynomial over the real axis. Since the series S_z converges to an inner analytic function $w(z)$, which also reduces to a real function on the interval $(-1, 1)$ of the real axis, it follows that $\gamma(z)$ reduces to a real function on the interval $(-1, 1)$ of the real axis as well. This establishes that the function $\gamma(z)$ has all the required properties and is therefore an inner analytic function.

Assuming that the series S_z is convergent, and thus that $w(z)$ has at most borderline hard singularities, this new series C_z generates a new inner analytic function $\gamma(z)$ that has only soft singularities and hence C_z converges absolutely and uniformly to a continuous function. From that function and the explicit poles we can then reconstruct the original function, in a piecewise fashion between pairs of adjacent dominant singularities. If the series S_z was already absolutely and uniformly convergent, the new center series C_z will allow one to take one more derivative, compared with the situation regarding S_z.

Note that between two adjacent singularities the function $w(z)$ is analytic over sections of the unit circle, and hence piecewise C^∞, so that this process can be taken, in principle, as far as one wishes, by the iteration of this procedure. In order to do this one has to re-examine the set of singularities of the resulting function $\gamma(z)$ because, with the softening of the dominant singularities, there may be now more singularities just as soft as those in the first set became. This will generate a new set of dominant singularities, and assuming that this set is also finite in number, one may iterate the procedure. The result will be a series that not only is absolutely and uniformly convergent, but is also one that can be differentiated one more time and still result in an equally convergent series.

Note also that we may as well start the process with a series that if flatly divergent, and that there is nothing to prevent us from recovering from it the original function, from which the coefficients were obtained. This can always be done, at least in principle, if the inner analytic function has at most a finite number of isolated singularities on the unit circle, each one with a finite degree of hardness. On the other hand, it cannot be done if there is an infinite number of dominant singularities, or if any individual singularity is an infinitely hard one, such as an essential singularity.

Arguably the most difficult step in this process is the determination of the singularities of the inner analytic function from the series. If there is enough information about the real function that originated it, or about the circumstances of the application involved, it may be possible to guess at the set of singularities. Otherwise, this information has to be obtained from the structure of the series itself.

However, this may not be such a grave difficulty as it might appear at first, since a given singularity structure characterizes a whole class of series, not a single series. For example, all series which have monotonic coefficients with step 1 have a single dominant singularity at $z = 1$. Series which have monotonic coefficients with step 1 and a factor of $(-1)^k$ added to the coefficients have a single dominant singularity at $z = -1$. Series with monotonic coefficients with step 2 have two dominant singularities, at $z = 1$ and at $z = -1$. Series with monotonic coefficients with step 2 and a factor of $(-1)^j$ added to the coefficients have two dominant singularities, at $z = \imath$ and at $z = -\imath$, and so on. In Appendix B we will give several simple examples of the construction of center series.

Observe also that this whole procedure is safe in the sense that if one guesses erroneously at the singularities, the worst that can happen is that no improvement in convergence is obtained. Besides, at least in principle the factorization process can be considered in reverse, in the sense that one may analyze the structure of the series S_z in order to discover what set of factors would do the trick of resulting in a series C_z with coefficients that go to zero faster than the original ones. If this problem is solvable, it in fact *determines* the location of the dominant singularities of the inner analytic function over the unit circle, by what turns out to have the nature of a purely algebraic method, leading to a polynomial $P_N(z)$ that implements the softening of the dominant singularities.

7 Conclusions

The sometimes complicated questions of convergence of Fourier series can be mapped onto the convergence of Taylor series of analytic functions. The modes of convergence of DP Fourier series can be classified according to the singularity structure of the corresponding inner analytic functions. The extreme cases of strong convergence and strong divergence are easily identified and classified, as was shown in the previous paper [1]. Simple tests can be used to identify these cases.

Weakly convergent DP Fourier series present a much more delicate and difficult problem. It can be shown that all these cases are translated, in the complex formalism, into the behavior of inner analytic functions and their Taylor series at the rim of the maximum disk of convergence of these series, which in this case is the unit circle. The treatment of this case required the introduction of a classification scheme for singularities of inner analytic functions. These were classified as either soft or hard, depending on the behavior of the inner analytic functions near them, and subsequently by integer indices giving the degrees of either softness or hardness of the singularities. Of particular importance are the degrees which we named as borderline soft and borderline hard.

This classification scheme led to the concept of integral-differential chains of inner analytic functions, which were needed in order to relate the classification of singularities with a corresponding classification of modes of convergence of the series associated to the functions. Definite results were obtained only for a certain subset of all possible such chains, consisting of regular chains in which the series $S_{v,h0}$ is an extended monotonic series. This defines a certain class of series and corresponding functions. With the limitation that we must stay within this class, it was shown that the existence and level of convergence of DP Fourier series is ruled by the nature of the dominant singularities of the inner analytic functions which are located at the rim of their maximum disk of convergence, which is the unit circle.

As a result of this classification, by constructing the inner analytic function of a given DP Fourier series in this class, one can determine the convergence of the series via the examination of the singularities of that function. At a very basic level, one can determine whether the series is strongly divergent or strongly convergent by simply determining the position of possible singularities of the inner analytic function. This leads to the three-pronged basic decision process: if there is a singularity of any type within the open unit disk, then the S_v series is divergent everywhere; if there are no singularities within the closed unit disk, then the S_v series converges everywhere to a C^∞ function; if there are one or more singularities on the unit circle, but none within the open unit disk, then the convergence of the S_v series is determined by the degree of hardness or of softness of the dominant singularities on the unit circle.

The last alternative leads to another three-pronged decision process, this time based on the type of the dominant singularities of the inner analytic function found on the unit circle. According to the underlying structure that was uncovered, at this finer level the decision structure leads to the following basic alternatives: if the dominant singularities are soft singularities, then the S_v series converges absolutely and uniformly everywhere; if they are borderline hard singularities, then the S_v series converges point-wise almost everywhere, but does not converge absolutely; if they are hard singularities with degree of hardness 1 or greater, then the series S_v diverges everywhere.

This classification scheme for all the modes of convergence, and of the corresponding degrees of softness or hardness of the dominant singularities, is illustrated in Table 1, which gives also some additional information. By and large, as the singularities become softer the series become more convergent over the unit circle, and converge to smoother functions. Given the convergence mode of the S_v series, one can then derive the corresponding mode for the DP Fourier series, which are also included in the table.

In terms of the analytic character of the limiting functions, at the most basic level strongly convergent DP Fourier series converge to restrictions to the whole unit circle of complex C^∞ analytic functions, while weakly convergent DP Fourier series converge to

Dominant Singularities	Convergence of S_v	Behavior of Coefficients	Convergence of $S_{\rm c}$ and $S_{\rm s}$	Character of $f(\theta)$
n-hard $n \geq 2$	divergent ew	$\|a_k\| \propto k^p$ $n-1 \leq p < n$	divergent aew	currently unknown
1-hard	divergent ew	$\|a_k\| \propto k^p$ $0 \leq p < 1$	divergent aew	δ-"function" for $p = 0$
borderline hard	point-wise aew	$\|a_k\| \propto 1/k^p$ $0 < p \leq 1$	point-wise aew	cont aew diff aew
borderline soft	absolute and uniform ew	$\|a_k\| \propto 1/k^p$ $1 < p \leq 2$	absolute and uniform ew	cont ew diff aew
1-soft	absolute and uniform ew	$\|a_k\| \propto 1/k^p$ $2 < p \leq 3$	absolute and uniform ew	diff ew C^2 aew
n-soft $n \geq 2$	absolute and uniform ew	$\|a_k\| \propto 1/k^p$ $n+1 < p \leq n+2$	absolute and uniform ew	C^n ew C^{n+1} aew

Table 1: A table showing the proposed classification of modes of convergence, singularity structure and limiting function properties, within the class of regular integral-differential chains in which the series $S_{v,h0}$ is an extended monotonic series. The abbreviation "ew" stands for "everywhere" and "aew" for "almost everywhere". The abbreviations "cont" and "diff" stand respectively for "continuous" and "differentiable". Integration goes downward through the lines, differentiation goes upward.

globally C^n functions which are also sectionally C^{n+1}, where n is the degree of softness of the dominant singularities on the unit circle. In the case in which the dominant singularities are borderline hard, the series converge to sectionally continuous and differentiable functions, which however are not globally continuous. In addition to this, in the case in which the dominant singularities are simple poles one may have representations of singular objects such as the Dirac delta "function", as was shown in the previous paper [1]. However, this last alternative has not yet been explored in much detail.

Moreover, we presented the process of singularity factorization, through which, given an arbitrary DP Fourier series, which can even be divergent almost everywhere, one can construct from it other expressions involving trigonometric series, that converge to the function that gave origin to the given DP Fourier series. This works by the construction of a new complex power series from the Taylor series S_z of the corresponding inner analytic function. We call these new series C_z the center series of the series S_z that converges to the inner analytic function associated to the original DP Fourier series. If the original series was very weakly convergent, then this new series will have much better convergence characteristics. Even if the original series is divergent one can still construct expressions involving center series that converge to the original function, in a piecewise fashion. In this way, a more practical means of recovery of the original function is provided, if compared to the explicit determination of the inner analytic function $w(z)$ in closed form, in order to enable one to take its limit to the unit circle explicitly.

Several points are left open and represent interesting possibilities for further development of the subject. One was presented in the previous paper [1] and consists of the question of whether or not there are real functions which generate strongly divergent DP Fourier series. The conjecture is that there are none, in which case taking limits of inner analytic functions from within the open unit disk would be established as a process for the generation, almost everywhere on the unit circle, of all real functions from which it is possible to define the coefficients of a DP Fourier series. Another interesting question, posed in this paper, is whether or not there are S_v series with coefficients a_k that behave as $|a_k| \propto 1/k^p$ with $0 < p \leq 1$ and that converge everywhere on the unit circle. If there are, it would be necessary to consider the extension of our classification scheme to other classes of DP Fourier series.

Since an absolutely and uniformly convergent DP Fourier series usually converges *much* faster than a non-absolutely and non-uniformly convergent one, doing the $S_v \to C_v$ transformation can be of enormous numerical advantage. One verifies that, the slower the convergence of S_v, caused by a value of p closer to zero when $a_k \propto 1/k^p$, the more advantageous is the use of the series C_v. Near the special points the gain is more limited, but it still exists. For simple well-known series such as the square wave, with coefficients that go to zero as $1/k$, which is the fastest possible approach within the $S_{v,h0}$ class, on average over the whole domain, and for the higher levels of numerical precision required of the results, the speedup can be as high as 1000 or more, as we will show elsewhere [3].

8 Acknowledgements

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A Appendix: Technical Proofs

A.1 Evaluations of Convergence

It is not a difficult task to establish the absolute and uniform convergence of DP Fourier series, or the lack thereof, starting from the behavior of the coefficients of the series in the limit $k \to \infty$, if we assume that they behave as inverse powers of k for large values of k. If we have a complex series S_v with coefficients a_k,

$$S_v = \sum_{k=1}^{\infty} a_k v^k,$$

where $|v| = 1$, then it is absolutely convergent if and only if the series $\overline{S}_v$ of the absolute values of the coefficients,

$$\begin{aligned} \overline{S}_v &= \sum_{k=1}^{\infty} |a_k||v|^k \\ &= \sum_{k=1}^{\infty} |a_k|, \end{aligned}$$

converges. One can show that this sum will be finite if, for k above a certain minimum value k_m, it holds that

$$|a_k| \leq \frac{A}{k^{1+\varepsilon}},$$

for some positive real constant A and some real constant $\varepsilon > 0$. This is true because the sum of a finite set of initial terms is necessarily finite, and because in this case we may bound the remaining infinite sum from above by a convergent asymptotic integral,

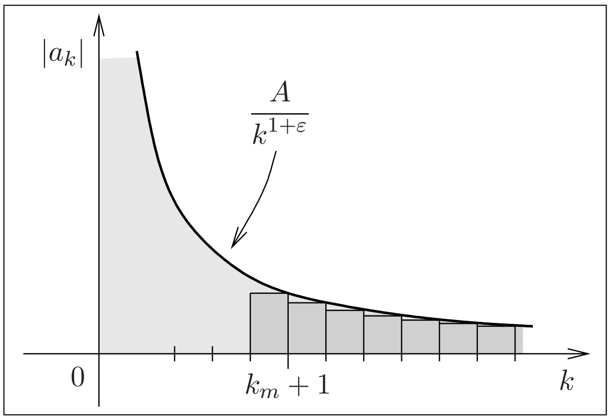

Figure 1: Illustration of the upper bounding of the sum of the coefficients $|a_k|$ by an integral.

$$\begin{aligned}\sum_{k=k_m+1}^{\infty}|a_k| &\leq \sum_{k=k_m+1}^{\infty}\frac{A}{k^{1+\varepsilon}}\\ &< \int_{k_m}^{\infty}dk\,\frac{A}{k^{1+\varepsilon}}\\ &= \frac{-A}{\varepsilon}\,\frac{1}{k^{\varepsilon}}\Big[_{k_m}^{\infty}\\ &= \frac{A}{\varepsilon}\,\frac{1}{k_m^{\varepsilon}},\end{aligned}$$

as illustrated in Figure 1. In that illustration each vertical rectangle has base 1 and height given by $|a_k|$, and therefore area given by $|a_k|$. As one can see, the construction is such that the set of all such rectangles is below the graph of the function $A/k^{1+\varepsilon}$, and therefore the sum of their areas is contained within the area under that graph, to the right of k_m. This establishes the necessary inequality between the sum and the integral.

So long as ε is not zero, this establishes an upper bound to a sum of positive quantities, which is therefore a monotonically increasing sum. It then follows from the well-known theorem of real analysis that the sum necessarily converges, and therefore the series S_v is absolutely convergent. The same is then true for the corresponding DP Fourier series.

In addition to this, one can see that the convergence condition does not depend on θ, since that dependence is only within the complex variable $v = \exp(\imath\theta)$, and vanishes when we take absolute values. This implies uniform convergence because, given a strictly positive real number ϵ, absolute convergence for this value of ϵ implies convergence for this same value of ϵ, with the same solution $k(\epsilon)$ for the convergence condition. This makes it clear that the solution of the convergence condition for k is independent of position and therefore that the series is also uniformly convergent. Once more, the same is then true for the corresponding DP Fourier series.

This establishes a sufficient condition for the absolute and uniform convergence of DP Fourier series. On the other hand, if we have that, for k above a certain minimum value k_m,

$$|a_k| \geq \frac{A}{k^{1-\varepsilon}},$$

with positive real A and real $\varepsilon \geq 0$, then it is possible to bound the sum $\overline{S}_v$ from below by an asymptotic integral that diverges to positive infinity. This is done in a way similar to

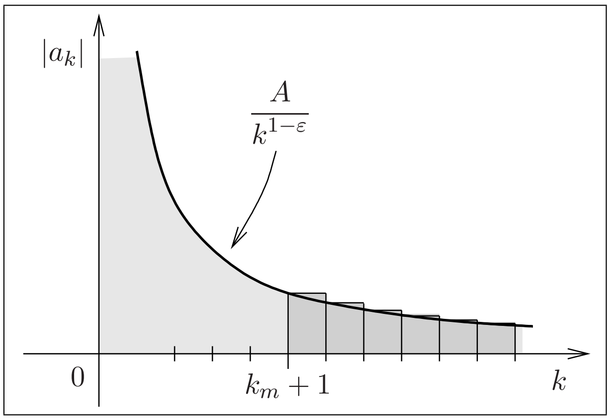

Figure 2: Illustration of the lower bounding of the sum of the coefficients $|a_k|$ by an integral.

the one used for the establishment of the upper bound, but inverting the situation so as to keep the area under the graph contained within the combined areas of the rectangles, as illustrated in Figure 2. The argument then establishes in this case that, for $\varepsilon > 0$

$$\begin{aligned}
\sum_{k=k_m+1}^{\infty} |a_k| &\geq \sum_{k=k_m+1}^{\infty} \frac{A}{k^{1-\varepsilon}} \\
&> \int_{k_m+1}^{\infty} dk\, \frac{A}{k^{1-\varepsilon}} \\
&= \frac{A}{\varepsilon}\, k^{\varepsilon} \Big[_{k_m+1}^{\infty} \\
&= -\frac{A}{\varepsilon}\,(k_m+1)^{\varepsilon} + \frac{A}{\varepsilon} \lim_{k\to\infty} k^{\varepsilon},
\end{aligned}$$

and therefore that $\overline{S}_v$ diverges to infinity. A similar calculation can be performed in the case $\varepsilon = 0$, leading to logarithms and yielding the same conclusions. This does not prove or disprove convergence itself, but it does establish the absence of absolute convergence. It also shows that, so long as $|a_k|$ behaves as a power of k for large k, the previous condition is both sufficient and necessary for absolute convergence.

A.2 Infinitely Soft Singularities

Consider the following power series, which has coefficients that converge monotonically to zero from positive values,

$$S_z = \sum_{k=1}^{\infty} \frac{1}{\mathrm{e}^{\sqrt{k}}}\, z^k.$$

If we apply the ratio test to it, we get

$$\begin{aligned}
R &= \frac{\mathrm{e}^{\sqrt{k}}|z|^{k+1}}{\mathrm{e}^{\sqrt{k+1}}|z|^{k}} \\
&= \mathrm{e}^{\sqrt{k}-\sqrt{k+1}}\rho \quad \Rightarrow \\
\ln(R) &= \ln(\rho) + \sqrt{k} - \sqrt{k+1},
\end{aligned}$$

where $|z| = \rho$. In the large-k limit we have

$$
\begin{aligned}
\lim_{k\to\infty}\ln(R) &= \ln(\rho)+\lim_{k\to\infty}\left(\sqrt{k}-\sqrt{k+1}\right)\\
&= \ln(\rho)+\lim_{k\to\infty}\left[\sqrt{k}\left(1-\sqrt{1+\frac{1}{k}}\right)\right]\\
&= \ln(\rho)+\lim_{k\to\infty}\left[\sqrt{k}\left(1-1-\frac{1}{2k}\right)\right]\\
&= \ln(\rho)-\lim_{k\to\infty}\frac{1}{2\sqrt{k}}\\
&= \ln(\rho).
\end{aligned}
$$

It follows that in the limit we have $R=\rho$ and therefore the conditions of the test are satisfied if and only if $\rho<1$. This establishes the open unit disk as the maximum disk of convergence of the series. Within this disk the series converges to an inner analytic function $w(z)$, and we may write

$$
w(z)=\sum_{k=1}^{\infty}\frac{1}{\mathrm{e}^{\sqrt{k}}}\,z^{k}.
$$

Since the maximum disk of convergence of the series is the open unit disk, this function must have at least one singularity on the unit circle. Note that since the series is monotonic with step 1, we already know that $w(z)$ has a single dominant singularity on that circle, located at $z=1$. Consider now the logarithmic derivatives of this function. Using the series we have within the open unit disk, for the $n^{\rm th}$ logarithmic derivative of $w(z)$,

$$
w^{n\bullet}(z)=\sum_{k=1}^{\infty}\frac{k^{n}}{\mathrm{e}^{\sqrt{k}}}\,z^{k}.
$$

This notation includes the original series as the case $n=0$. All these series converge on the open unit disk, of course. Let us now consider the corresponding series of absolute values, for $\rho=1$,

$$
\overline{S}_{v}^{n\bullet}=\sum_{k=1}^{\infty}\frac{k^{n}}{\mathrm{e}^{\sqrt{k}}}.
$$

The terms of this sum can be bounded, for $k>k_m$ and some minimum value k_m of k, by the function $1/k^2$, since we have that it is always possible to find a value ξ_m of a variable ξ such that for $\xi>\xi_m$ we have

$$
\mathrm{e}^{-\xi}<\frac{1}{\xi^{2n+4}},
$$

since the exponential goes to zero faster than any inverse power. Making $\xi^2=k$ and therefore $\xi=\sqrt{k}$ we have

$$
\begin{aligned}
\mathrm{e}^{-\sqrt{k}} &< \frac{1}{k^{n+2}} \quad\Rightarrow\\
\frac{k^{n}}{\mathrm{e}^{\sqrt{k}}} &< \frac{1}{k^{2}},
\end{aligned}
$$

thus proving the assertion. This implies that all these series are absolutely and uniformly convergent over the whole unit circle, to continuous functions. Therefore, all these series must have a soft singularity on the unit circle, at $z = 1$. This is one example in which we may differentiate as many times as we will, without the singularity ever becoming hard. Therefore, that singularity is necessarily an infinitely soft one.

Note that in this case the DP Fourier series on the unit circle converge to C^∞ functions, although there are singularities on that circle. Although these real functions are C^∞, in the real sense of this concept, the complex function $w(z)$ cannot be C^∞ on the unit circle, in the complex sense of the concept. One may ask how can a restriction of the complex function $w(z)$, which is C^∞ in the whole interior of the unit disk, be C^∞ at the boundary of the disk while $w(z)$ itself is not.

The answer is that the C^∞ condition in the real sense is a weaker condition than the C^∞ condition in the complex sense. While in the case of the real functions on the unit circle only the derivatives with respect to θ must exist, in the case of the complex function the derivatives in the perpendicular direction, that is those with respect to ρ, must also exist, and in fact must give the same values as the derivatives in the direction of θ.

Unlike real functions over one-dimensional domains, which can be folded around at will, complex analytic functions over two-dimensional domains are rigid objects. If one restricts such a function to a one-dimensional domain and then folds that domain around, the resulting real function over it may no longer be the restriction of a complex function to the new one-dimensional domain resulting from the folding process.

B Appendix: Examples of Center Series

In this appendix we will give a few simple illustrative examples of the construction of center series. In many cases the construction of a center series constitutes a practical way to determine the corresponding real function, and to thus take advantage of the very existence of the inner analytic function, for example when it is not possible to exhibit that function in closed form, in order to explicitly take its limit to the unit circle.

The process of construction consists of three parts, starting with the determination of the power series S_z from the original DP Fourier series, which is simple and can always be done without any difficulty. The second step is the construction of the complex center series C_z, which is operationally fairly simple but depends on the knowledge of the complete set of dominant singularities over the unit circle of the inner analytic function $w(z)$ that the series S_z converges to.

The last step is the recovery from S_z written in terms of C_z of the real and imaginary parts of $w(z)$, in order to obtain the center series versions of the original DP Fourier series and of its FC series. This is straightforward but can become, in some cases, a rather long algebraic process. In each example we will develop explicitly all these steps, with a reasonable amount of detail.

Some of the examples that follow are the same that were worked out by another method in the appendices of the already mentioned previous paper [1]. They are presented in the same order as in that paper. It is understood that the final forms obtained for the functions $f_{\rm s}(\theta)$ and $f_{\rm c}(\theta)$ in terms of the center series are valid only away from the special points.

B.1 A Regular Sine Series with All k

Consider the Fourier series of the one-cycle unit-amplitude sawtooth wave, which is just the linear function θ/π between $-\pi$ and π. As is well known it is given by the sine series

$$S_{\rm s} = -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\,\sin(k\theta).$$

The corresponding FC series is then

$$\bar{S}_{\rm s} = -\,\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,\cos(k\theta),$$

the complex S_v series is given by

$$S_v = -\,\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,v^{k},$$

and the complex power series S_z is given by

$$S_z = -\,\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,z^{k}.$$

The ratio test tells us that the disk of convergence of S_z is the unit disk. If we consider the inner analytic function $w(z)$ within this disk we observe that $w(0) = 0$, as expected. We have for this function

$$w(z) = -\,\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,z^{k}.$$

Being given by a monotonic series of step 1 modified by the factor of $(-1)^{k}$, this function has a single dominant singularity at $z = -1$, where it diverges to infinity, as one can easily verify

$$\begin{aligned} w(-1) &= -\,\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k} \\ &\rightarrow -\infty. \end{aligned}$$

We must therefore use a single factor of $(z + 1)$ in the construction of the center series,

$$\begin{aligned} C_z &= -\,\frac{2}{\pi}\,(z+1)\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,z^{k} \\ &= -\,\frac{2}{\pi}\left[\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,z^{k+1} + \sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\,z^{k}\right] \\ &= -\,\frac{2}{\pi}\left[\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k-1}\,z^{k} - z + \sum_{k=2}^{\infty}\frac{(-1)^{k}}{k}\,z^{k}\right] \\ &= -\,\frac{2}{\pi}\left[-z - \sum_{k=2}^{\infty}(-1)^{k}\left(\frac{1}{k-1} - \frac{1}{k}\right)z^{k}\right] \\ &= \frac{2}{\pi}\left[z + \sum_{k=2}^{\infty}\frac{(-1)^{k}}{(k-1)k}\,z^{k}\right] \\ &= \frac{2}{\pi}\left[z + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\,z^{k+1}\right] \\ &= \frac{2}{\pi}\,z\left[1 - \sum_{k=1}^{\infty}\frac{(-1)^{k}}{k(k+1)}\,z^{k}\right]. \end{aligned}$$

Unlike the original series, with coefficients that behave as $1/k$, this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore is absolutely and uniformly convergent to a continuous function. This shows, in particular, that our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{2}{\pi}\,\frac{z}{z+1}\left[1-\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,z^k\right],$$

with the singularity factored out and where the series involved is absolutely and uniformly convergent, and therefore converges much faster than the original one.

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. We have on the unit circle

$$\begin{aligned}\frac{z}{z+1} &= \frac{z(z^*+1)}{(z+1)(z^*+1)}\\ &= \frac{1+z}{2+z+z^*}\\ &= \frac{1+\cos(\theta)+\imath\sin(\theta)}{2+2\cos(\theta)}\\ &= \frac{1}{2}+\frac{\imath}{2}\,\frac{\sin(\theta)}{1+\cos(\theta)}.\end{aligned}$$

If we write this in terms of $\theta/2$ we get

$$\begin{aligned}\frac{z}{z+1} &= \frac{1}{2}+\frac{\imath}{2}\,\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}\\ &= \frac{1}{2}+\frac{\imath}{2}\,\frac{\sin(\theta/2)}{\cos(\theta/2)},\end{aligned}$$

and therefore

$$\begin{aligned}S_v &= \frac{1}{\pi}\left[1+\imath\,\frac{\sin(\theta/2)}{\cos(\theta/2)}\right]\left[1-\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\cos(k\theta)-\imath\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\sin(k\theta)\right]\\
&= \frac{1}{\pi}\left\{\left[1-\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\cos(k\theta)\right]+\frac{\sin(\theta/2)}{\cos(\theta/2)}\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\sin(k\theta)\right]\right\}+\\
&\quad+\imath\,\frac{1}{\pi}\left\{\frac{\sin(\theta/2)}{\cos(\theta/2)}\left[1-\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\cos(k\theta)\right]-\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\sin(k\theta)\right]\right\}\\
&= \frac{1}{\pi}\left\{1-\frac{1}{\cos(\theta/2)}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\left[\cos\left(\frac{\theta}{2}\right)\cos(k\theta)-\sin\left(\frac{\theta}{2}\right)\sin(k\theta)\right]\right\}+\\
&\quad+\imath\,\frac{1}{\pi}\left\{\frac{\sin(\theta/2)}{\cos(\theta/2)}-\frac{1}{\cos(\theta/2)}\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\left[\sin\left(\frac{\theta}{2}\right)\cos(k\theta)+\cos\left(\frac{\theta}{2}\right)\sin(k\theta)\right]\right\}\\
&= \frac{1}{\pi\cos(\theta/2)}\left[\cos(\theta/2)-\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\cos\left(\frac{2k+1}{2}\,\theta\right)\right]+\\
&\quad+\imath\,\frac{1}{\pi\cos(\theta/2)}\left[\sin(\theta/2)-\sum_{k=1}^{\infty}\frac{(-1)^k}{k(k+1)}\,\sin\left(\frac{2k+1}{2}\,\theta\right)\right].\end{aligned}$$

The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta)=\frac{1}{\pi\cos(\theta/2)}\left[\sin(\theta/2)-\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k(k+1)}\,\sin\!\left(\frac{2k+1}{2}\,\theta\right)\right],$$

and the corresponding FC function $f_{\rm c}(\theta)=\bar{f}_{\rm s}(\theta)$ is given by the real part,

$$f_{\rm c}(\theta)=\frac{1}{\pi\cos(\theta/2)}\left[\cos(\theta/2)-\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k(k+1)}\,\cos\!\left(\frac{2k+1}{2}\,\theta\right)\right].$$

Both of these series are absolutely and uniformly convergent.

B.2 A Regular Sine Series with Odd k

Consider the Fourier series of the standard unit-amplitude square wave. As is well known it is given by the sine series

$$S_{\rm s}=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,\sin[(2j+1)\theta].$$

The corresponding FC series is then

$$\bar{S}_{\rm s}=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,\cos[(2j+1)\theta],$$

the complex S_v series is given by

$$S_v=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,v^{2j+1},$$

and the complex power series S_z is given by

$$S_z=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+1}.$$

The ratio test tells us that the disk of convergence of S_z is the unit disk. If we consider the inner analytic function $w(z)$ within this disk we observe that $w(0)=0$, as expected. We have for this function

$$w(z)=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+1}.$$

Being given by a monotonic series of step 2 this function has two dominant singularities, located at $z=1$ and at $z=-1$, where it diverges to infinity, as one can easily verify,

$$\begin{aligned}
w(1) &= \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\\
&\rightarrow \infty,\\
w(-1) &= -\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\\
&\rightarrow -\infty.
\end{aligned}$$

We must therefore use the two factors $(z-1)(z+1) = z^2 - 1$ in the construction of the center series,

$$\begin{aligned}
C_z &= \frac{4}{\pi}\left(z^2-1\right)\sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+1} \\
&= \frac{4}{\pi}\left(\sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+3} - \sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+1}\right) \\
&= \frac{4}{\pi}\left(\sum_{j=1}^{\infty}\frac{1}{2j-1}\,z^{2j+1} - z - \sum_{j=1}^{\infty}\frac{1}{2j+1}\,z^{2j+1}\right) \\
&= \frac{4}{\pi}\left[-z + \sum_{j=1}^{\infty}\left(\frac{1}{2j-1} - \frac{1}{2j+1}\right)z^{2j+1}\right] \\
&= \frac{4}{\pi}\left(-z + \sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,z^{2j+1}\right) \\
&= \frac{4}{\pi}\,z\left(-1 + \sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,z^{2j}\right).
\end{aligned}$$

Unlike the original series, with coefficients that behave as $1/k$ (with $k = 2j + 1$), this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore is absolutely and uniformly convergent to a continuous function. This shows, in particular, that our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{4}{\pi}\,\frac{z}{z^2-1}\left(-1 + \sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,z^{2j}\right),$$

with the singularities factored out and where the series involved is absolutely and uniformly convergent, and therefore converges much faster than the original one.

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. We have on the unit circle

$$\begin{aligned}
\frac{z}{z^2-1} &= \frac{z\left[(z^*)^2-1\right]}{(z^2-1)\left[(z^*)^2-1\right]} \\
&= \frac{z^*-z}{2-z^2-(z^*)^2} \\
&= \frac{-2\imath\sin(\theta)}{2-2\cos(2\theta)} \\
&= \frac{-\imath\sin(\theta)}{1-\cos^2(\theta)+\sin^2(\theta)} \\
&= \frac{-\imath\sin(\theta)}{2\sin^2(\theta)} \\
&= \frac{-\imath}{2\sin(\theta)},
\end{aligned}$$

and therefore

$$\begin{aligned} S_v &= \frac{4}{\pi}\,\frac{-\imath}{2\sin(\theta)}\left[-1+\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,\cos(2j\theta)+\imath\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,\sin(2j\theta)\right]\\ &= \frac{2}{\pi\sin(\theta)}\left[\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,\sin(2j\theta)\right]+\imath\,\frac{2}{\pi\sin(\theta)}\left[1-\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,\cos(2j\theta)\right]. \end{aligned}$$

The original DP function is given by the imaginary part,

$$f_{\rm s}(\theta)=\frac{2}{\pi\sin(\theta)}\left[1-\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,\cos(2j\theta)\right],$$

and the corresponding FC function $f_{\rm c}(\theta)=\bar{f}_{\rm s}(\theta)$ is given by the real part,

$$f_{\rm c}(\theta)=\frac{2}{\pi\sin(\theta)}\left[\sum_{j=1}^{\infty}\frac{2}{4j^2-1}\,\sin(2j\theta)\right].$$

Both of these series are absolutely and uniformly convergent.

B.3 A Regular Sine Series with Even k

Consider the Fourier series of the two-cycle unit-amplitude sawtooth wave. As is well known it is given by the sine series

$$S_{\rm s}=-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j}\,\sin[(2j)\theta].$$

The corresponding FC series is then

$$\bar{S}_{\rm s}=-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j}\,\cos[(2j)\theta],$$

the complex S_v series is given by

$$S_v=-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j}\,v^{2j},$$

and the complex power series S_z is given by

$$S_z=-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j}\,z^{2j}.$$

The ratio test tells us that the disk of convergence of S_z is the unit disk. If we consider the inner analytic function $w(z)$ within this disk we observe that $w(0)=0$, as expected. We have for this function

$$w(z)=-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j}\,z^{2j}.$$

Being given by a monotonic series of step 2 this function has two dominant singularities, located at $z = 1$ and at $z = -1$, where it diverges to infinity, as one can easily verify,

$$\begin{aligned} w(1) &= -\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j} \\ &\rightarrow -\infty, \\ w(-1) &= -\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{2j} \\ &\rightarrow -\infty. \end{aligned}$$

We must therefore use the two factors $(z-1)(z+1) = z^2 - 1$ in the construction of the center series,

$$\begin{aligned} C_z &= -\frac{4}{\pi}\left(z^2-1\right)\sum_{j=1}^{\infty}\frac{1}{2j}\,z^{2j} \\ &= -\frac{4}{\pi}\left(\sum_{j=1}^{\infty}\frac{1}{2j}\,z^{2j+2} - \sum_{j=1}^{\infty}\frac{1}{2j}\,z^{2j}\right) \\ &= -\frac{4}{\pi}\left(\sum_{j=2}^{\infty}\frac{1}{2j-2}\,z^{2j} - \frac{z^2}{2} - \sum_{j=2}^{\infty}\frac{1}{2j}\,z^{2j}\right) \\ &= -\frac{2}{\pi}\left[-z^2 + \sum_{j=2}^{\infty}\left(\frac{1}{j-1}-\frac{1}{j}\right)z^{2j}\right] \\ &= \frac{2}{\pi}\left[z^2 - z^2\sum_{j=2}^{\infty}\frac{1}{(j-1)j}\,z^{2j-2}\right] \\ &= \frac{2}{\pi}\,z^2\left[1 - \sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,z^{2j}\right]. \end{aligned}$$

Unlike the original series, with coefficients that behave as $1/k$ (with $k = 2j$), this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore is absolutely and uniformly convergent to a continuous function. This shows, in particular, that our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$S_z = \frac{2}{\pi}\,\frac{z^2}{z^2-1}\left[1 - \sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,z^{2j}\right],$$

with the singularities factored out and where the series involved is absolutely and uniformly convergent, and therefore converges much faster than the original one.

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. We have on the unit circle

$$\frac{z^2}{z^2-1} = \frac{z^2\left[(z^*)^2-1\right]}{\left(z^2-1\right)\left[(z^*)^2-1\right]}$$

$$
\begin{aligned}
&= \frac{1-z^2}{2-z^2-(z^*)^2}\\
&= \frac{1-\cos(2\theta)-\imath\sin(2\theta)}{2-2\cos(2\theta)}\\
&= \frac{1}{2}-\frac{\imath}{2}\,\frac{\sin(2\theta)}{1-\cos(2\theta)}\\
&= \frac{1}{2}-\frac{\imath}{2}\,\frac{2\sin(\theta)\cos(\theta)}{2\sin^2(\theta)}\\
&= \frac{1}{2}-\frac{\imath}{2}\,\frac{\cos(\theta)}{\sin(\theta)},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
S_v &= \frac{1}{\pi}\left[1-\imath\frac{\cos(\theta)}{\sin(\theta)}\right]\left[1-\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\cos(2j\theta)-\imath\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\sin(2j\theta)\right]\\
&= \frac{1}{\pi}\left\{\left[1-\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\cos(2j\theta)\right]-\frac{\cos(\theta)}{\sin(\theta)}\left[\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\sin(2j\theta)\right]\right\}+\\
&\quad +\imath\,\frac{1}{\pi}\left\{-\frac{\cos(\theta)}{\sin(\theta)}\left[1-\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\cos(2j\theta)\right]-\left[\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\sin(2j\theta)\right]\right\}\\
&= \frac{1}{\pi}\left\{1-\frac{1}{\sin(\theta)}\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,[\sin(\theta)\cos(2j\theta)+\cos(\theta)\sin(2j\theta)]\right\}+\\
&\quad +\imath\,\frac{1}{\pi}\left\{-\frac{\cos(\theta)}{\sin(\theta)}+\frac{1}{\sin(\theta)}\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,[\cos(\theta)\cos(2j\theta)-\sin(\theta)\sin(2j\theta)]\right\}\\
&= \frac{1}{\pi\sin(\theta)}\left\{\sin(\theta)-\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\sin[(2j+1)\theta]\right\}+\\
&\quad +\imath\,\frac{1}{\pi\sin(\theta)}\left\{-\cos(\theta)+\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\cos[(2j+1)\theta]\right\}.
\end{aligned}
$$

The original DP function is given by the imaginary part,

$$
f_{\rm s}(\theta) = -\frac{1}{\pi\sin(\theta)}\left\{\cos(\theta)-\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\cos[(2j+1)\theta]\right\},
$$

and the corresponding FC function $f_{\rm c}(\theta)=\bar{f}_{\rm s}(\theta)$ is given by the real part,

$$
f_{\rm c}(\theta) = \frac{1}{\pi\sin(\theta)}\left\{\sin(\theta)-\sum_{j=1}^{\infty}\frac{1}{j(j+1)}\,\sin[(2j+1)\theta]\right\}.
$$

Both of these series are absolutely and uniformly convergent.

B.4 A Regular Cosine Series with Odd k

Consider the Fourier series of the unit-amplitude triangular wave. As is well known it is given by the cosine series

$$S_{\rm c} = -\,\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,\cos[(2j+1)\theta].$$

The corresponding FC series is then

$$\bar{S}_{\rm c} = -\,\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,\sin[(2j+1)\theta].$$

Note that due to the factors of $1/k^2$ (with $k=2j+1$), these series are already absolutely and uniformly convergent. But we will proceed with the construction in any case. The complex S_v series is given by

$$S_v = -\,\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,v^{2j+1},$$

and the complex power series S_z is given by

$$S_z = -\,\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+1}.$$

The ratio test tells us that the disk of convergence of S_z is the unit disk. If we consider the inner analytic function $w(z)$ within this disk we observe that $w(0)=0$, as expected. We have for this function

$$w(z) = -\,\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+1}.$$

Being given by a monotonic series of step 2 this function has two dominant singularities, located at $z=1$ and at $z=-1$, as one can easily verify by taking its logarithmic derivative, which is proportional to the inner analytic function of the square wave, that we examined before in Subsection B.2,

$$z\,\frac{dw(z)}{dz} = -\,\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+1}.$$

We must therefore use the two factors $(z-1)(z+1)=z^2-1$ in the construction of the center series,

$$\begin{aligned}
C_z &= -\,\frac{8}{\pi^2}\left(z^2-1\right)\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+1}\\
&= -\,\frac{8}{\pi^2}\left[\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+3}-\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+1}\right]\\
&= -\,\frac{8}{\pi^2}\left[\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\,z^{2j+1}-z-\sum_{j=1}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{8}{\pi^2}\left\{-z+\sum_{j=1}^{\infty}\left[\frac{1}{(2j-1)^2}-\frac{1}{(2j+1)^2}\right]z^{2j+1}\right\} \\
&= \frac{8}{\pi^2}\left[z-z\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,z^{2j}\right] \\
&= \frac{8}{\pi^2}\,z\left[1-\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,z^{2j}\right].
\end{aligned}
$$

Unlike the original series, with coefficients that behave as $1/k^2$ (with $k = 2j + 1$), this series has coefficients that go to zero as $1/k^3$ when $k \to \infty$, and therefore converges faster than the original one. This shows, in particular, that our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$
S_z = \frac{8}{\pi^2}\,\frac{z}{z^2-1}\left[1-\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,z^{2j}\right],
$$

with the singularities factored out. Although both this series and the original one are absolutely and uniformly convergent, this converges faster, and may be differentiated once, still resulting in another series which is also absolutely and uniformly convergent. Note that in this case, as was discussed in the appendices of the previous paper [1], we are not able to write an explicit expression for $w(z)$ in terms of elementary function, so that we cannot explicitly take its limit to the unit circle. However, as one can see here we are still able to write a series to represent it over the unit circle, which converges faster that the original one. This gives us the possibility of calculating the function to any required precision level, and to do so efficiently.

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. We have on the unit circle, as we saw before in Subsection B.2,

$$
\frac{z}{z^2-1} = \frac{-\imath}{2\sin(\theta)},
$$

and therefore

$$
\begin{aligned}
S_v &= \frac{4}{\pi^2}\,\frac{-\imath}{\sin(\theta)}\left[1-\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,\cos(2j\theta)-\imath\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,\sin(2j\theta)\right] \\
&= \frac{4}{\pi^2\sin(\theta)}\left[-\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,\sin(2j\theta)\right]+ \\
&\quad +\imath\,\frac{4}{\pi^2\sin(\theta)}\left[-1+\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,\cos(2j\theta)\right].
\end{aligned}
$$

The original DP function is given by the real part,

$$
f_{\rm c}(\theta) = -\frac{4}{\pi^2\sin(\theta)}\left[\sum_{j=1}^{\infty}\frac{8j}{\left(4j^2-1\right)^2}\,\sin(2j\theta)\right],
$$

and the corresponding FC function $f_{\rm s}(\theta) = \bar{f}_{\rm c}(\theta)$ is given by the imaginary part,

$$f_{\rm s}(\theta) = -\frac{4}{\pi^2\sin(\theta)}\left[1 - \sum_{j=1}^{\infty}\frac{8j}{(4j^2-1)^2}\cos(2j\theta)\right].$$

Both of these series converge faster than the original Fourier series.

B.5 A Singular Cosine Series

Consider the Fourier series of the Dirac delta "function" centered at $\theta = \theta_1$, which we denote by $\delta(\theta - \theta_1)$. We may easily calculate its Fourier coefficients using the rules of manipulation of $\delta(\theta - \theta_1)$, thus obtaining $\alpha_k = \cos(k\theta_1)/\pi$ and $\beta_k = \sin(k\theta_1)/\pi$ for all k. The series is therefore the complete Fourier series given by

$$\begin{aligned} S_{\rm c} &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}[\cos(k\theta_1)\cos(k\theta) + \sin(k\theta_1)\sin(k\theta)] \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos(k\Delta\theta), \end{aligned}$$

where $\Delta\theta = \theta - \theta_1$. Apart from the constant term this is in fact a DP cosine series on this new variable. Clearly, this series diverges at all points in the interval $[-\pi, \pi]$. Undaunted by this, we proceed to construct the FC series, with respect to the new variable $\Delta\theta$, which turns out to be

$$\bar{S}_{\rm c} = \frac{1}{\pi}\sum_{k=1}^{\infty}\sin(k\Delta\theta),$$

a series that is also divergent, this time almost everywhere. If we define $v = \exp(\imath\theta)$ and $v_1 = \exp(\imath\theta_1)$ the corresponding complex series S_v is then given by

$$S_v = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{v}{v_1}\right)^k,$$

where we included the $k = 0$ term, and the corresponding complex power series S_z is given by

$$S_z = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{z}{z_1}\right)^k,$$

where $z = \rho v$ and $z_1 = v_1$ is a point over the unit circle. The ratio test tells us that the disk of convergence of S_z is the unit disk. This converges to a perfectly well-defined analytic function strictly inside the open unit disk. If we eliminate the constant term we get a series S'_z which converges to an inner analytic function rotated by the angle θ_1,

$$S'_z = \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{z}{z_1}\right)^k.$$

The dominant singularity is clearly at the point z_1, so we must use the factor $(z - z_1)$ in the construction of the corresponding center series,

$$
\begin{aligned}
C_z' &= (z - z_1) S_z' \\
&= \frac{1}{\pi} z_1 \left(\frac{z}{z_1} - 1 \right) \sum_{k=1}^{\infty} \left(\frac{z}{z_1} \right)^k \\
&= \frac{1}{\pi} z_1 \left[\sum_{k=1}^{\infty} \left(\frac{z}{z_1} \right)^{k+1} - \sum_{k=1}^{\infty} \left(\frac{z}{z_1} \right)^k \right] \\
&= \frac{1}{\pi} z_1 \left[\sum_{k=2}^{\infty} \left(\frac{z}{z_1} \right)^k - \left(\frac{z}{z_1} \right) - \sum_{k=2}^{\infty} \left(\frac{z}{z_1} \right)^k \right] \\
&= -\frac{1}{\pi} z.
\end{aligned}
$$

So we see that we get a remarkably simple result, since the center series can actually be added up exactly. We get therefore for the series S_z'

$$
S_z' = -\frac{1}{\pi} \frac{z}{z - z_1},
$$

and for the series S_z

$$
S_z = \frac{1}{2\pi} - \frac{1}{\pi} \frac{z}{z - z_1}.
$$

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. The explanation of the reasons why this is a representation of the Dirac delta "function" requires taking limits to the unit circle carefully, and since they were given in the previous paper [1], they will not be repeated here. We have, for z on the unit circle, so long as $\Delta\theta \neq 0$,

$$
\begin{aligned}
\frac{z}{z - z_1} &= \frac{z(z^* - z_1^*)}{(z - z_1)(z^* - z_1^*)} \\
&= \frac{1 - z z_1^*}{2 - z z_1^* - z^* z_1} \\
&= \frac{1 - \cos(\Delta\theta) - \imath \sin(\Delta\theta)}{2 - 2\cos(\Delta\theta)} \\
&= \frac{1}{2} - \imath \frac{1}{2} \frac{\sin(\Delta\theta)}{1 - \cos(\Delta\theta)}, \\
&= \frac{1}{2} - \imath \frac{1}{2} \frac{1 + \cos(\Delta\theta)}{\sin(\Delta\theta)},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
S_v &= \frac{1}{2\pi} - \frac{1}{2\pi} + \imath \frac{1}{2\pi} \frac{1 + \cos(\Delta\theta)}{\sin(\Delta\theta)} \\
&= \imath \frac{1}{2\pi} \frac{1 + \cos(\Delta\theta)}{\sin(\Delta\theta)}.
\end{aligned}
$$

The original DP "function" is given by the real part, and therefore we get

$$
f_{\mathrm{c}}(\theta) = 0,
$$

which is the correct value for the Dirac delta "function" away from the singular point at $\Delta\theta = 0$, and the corresponding FC function $f_{\mathrm{s}}(\theta) = \bar{f}_{\mathrm{c}}(\theta)$ is given by the imaginary part,

$$
f_{\mathrm{s}}(\theta) = \frac{1}{2\pi} \frac{1 + \cos(\Delta\theta)}{\sin(\Delta\theta)},
$$

which is the same result we obtained in the previous paper [1].

B.6 Another Regular Cosine Series with Odd k

Consider the Fourier series of the unit-amplitude square wave, shifted along the θ axis to θ', with $\theta - \theta' = \pi/2$, so that it becomes an even function. As is well known it is given by the cosine series

$$S_{\rm c} = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\cos[(2j+1)\theta],$$

where we have dropped the prime. The corresponding FC series is then

$$\bar{S}_{\rm c} = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\sin[(2j+1)\theta],$$

the complex S_v series is given by

$$S_v = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}v^{2j+1},$$

and the complex power series S_z is given by

$$S_z = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}z^{2j+1}.$$

The ratio test tells us that the disk of convergence of S_z is the unit disk. If we consider the inner analytic function $w(z)$ within this disk we observe that $w(0) = 0$, as expected. We have for this function

$$w(z) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}z^{2j+1}.$$

Being given by a monotonic series of step 2 modified by the factor of $(-1)^j$, this function has two dominant singularities, located at $z = \boldsymbol{\imath}$ and at $z = -\boldsymbol{\imath}$, where it diverges to infinity, as one can easily verify,

$$\begin{aligned} w(\boldsymbol{\imath}) &= \frac{4\boldsymbol{\imath}}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1} \\ &\rightarrow \boldsymbol{\imath}\infty, \\ w(-\boldsymbol{\imath}) &= -\frac{4\boldsymbol{\imath}}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1} \\ &\rightarrow -\boldsymbol{\imath}\infty. \end{aligned}$$

We must therefore use the two factors $(z - \boldsymbol{\imath})(z + \boldsymbol{\imath}) = z^2 + 1$ in the construction of the center series,

$$\begin{aligned} C_z &= \frac{4}{\pi}\left(z^2+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}z^{2j+1} \\ &= \frac{4}{\pi}\left[\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}z^{2j+3} + \sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}z^{2j+1}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{\pi}\left[\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{2j-1}\,z^{2j+1}+z+\sum_{j=1}^{\infty}\frac{(-1)^{j}}{2j+1}\,z^{2j+1}\right]\\
&= \frac{4}{\pi}\left[z-\sum_{j=1}^{\infty}(-1)^{j}\left(\frac{1}{2j-1}-\frac{1}{2j+1}\right)z^{2j+1}\right]\\
&= \frac{4}{\pi}\left[z-\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\,z^{2j+1}\right]\\
&= \frac{4}{\pi}\,z\left[1-\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\,z^{2j}\right].
\end{aligned}
$$

Unlike the original series, with coefficients that behave as $1/k$ (with $k = 2j + 1$), this series has coefficients that go to zero as $1/k^2$ when $k \to \infty$, and therefore is absolutely and uniformly convergent to a continuous function. This shows, in particular, that our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$
S_z = \frac{4}{\pi}\,\frac{z}{z^2+1}\left[1-\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\,z^{2j}\right],
$$

with the singularities factored out and where the series involved is absolutely and uniformly convergent, and therefore converges much faster than the original one.

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. We have on the unit circle

$$
\begin{aligned}
\frac{z}{z^2+1} &= \frac{z\left[(z^*)^2+1\right]}{(z^2+1)\left[(z^*)^2+1\right]}\\
&= \frac{z^*+z}{2+z^2+(z^*)^2}\\
&= \frac{2\cos(\theta)}{2+2\cos(2\theta)}\\
&= \frac{\cos(\theta)}{1+\cos^2(\theta)-\sin^2(\theta)}\\
&= \frac{\cos(\theta)}{2\cos^2(\theta)}\\
&= \frac{1}{2\cos(\theta)},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
S_v &= \frac{2}{\pi\cos(\theta)}\left[1-\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\cos(2j\theta)-\imath\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\sin(2j\theta)\right]\\
&= \frac{2}{\pi\cos(\theta)}\left[1-\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\cos(2j\theta)\right]+\imath\,\frac{2}{\pi\cos(\theta)}\left[-\sum_{j=1}^{\infty}\frac{2(-1)^{j}}{4j^2-1}\sin(2j\theta)\right].
\end{aligned}
$$

The original DP function is given by the real part,

$$f_{\rm c}(\theta) = \frac{2}{\pi\cos(\theta)}\left[1 - \sum_{j=1}^{\infty}\frac{2(-1)^j}{4j^2-1}\cos(2j\theta)\right],$$

and the corresponding FC function $f_{\rm s}(\theta) = \bar{f}_{\rm c}(\theta)$ is given by the imaginary part,

$$f_{\rm s}(\theta) = -\frac{2}{\pi\cos(\theta)}\left[\sum_{j=1}^{\infty}\frac{2(-1)^j}{4j^2-1}\sin(2j\theta)\right].$$

Both of these series are absolutely and uniformly convergent.

B.7 Another Regular Sine Series with Odd k

Consider the Fourier series of a periodic function built with segments of quadratic functions, joined together so that the resulting function is continuous and differentiable. As is well known it is given by the sine series

$$S_{\rm s} = \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\sin[(2j+1)\theta].$$

The corresponding FC series is then

$$\bar{S}_{\rm s} = \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\cos[(2j+1)\theta].$$

Note that due to the factors of $1/k^3$ (with $k = 2j+1$), these series are already absolutely and uniformly convergent. But we will proceed with the construction in any case. The complex S_v series is given by

$$S_v = \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}v^{2j+1},$$

and the complex power series S_z is given by

$$S_z = \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}z^{2j+1}.$$

The ratio test tells us that the disk of convergence of S_z is the unit disk. If we consider the inner analytic function $w(z)$ within this disk we observe that $w(0) = 0$, as expected. We have for this function

$$w(z) = \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}z^{2j+1}.$$

Being given by a monotonic series of step 2 this function has two dominant singularities, located at $z = 1$ and at $z = -1$, as one can easily verify by taking its second logarithmic derivative, which is proportional to the inner analytic function of the standard square wave, that we examined before in Subsection B.2,

$$
\begin{aligned}
z\,\frac{dw(z)}{dz} &= \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\,z^{2j+1},\\
z\,\frac{d}{dz}\left[z\,\frac{dw(z)}{dz}\right] &= \frac{32}{\pi^3}\sum_{j=0}^{\infty}\frac{1}{2j+1}\,z^{2j+1}.
\end{aligned}
$$

We must therefore use the two factors $(z-1)(z+1) = z^2-1$ in the construction of the center series,

$$
\begin{aligned}
C_z &= \frac{32}{\pi^3}\left(z^2-1\right)\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\,z^{2j+1}\\
&= \frac{32}{\pi^3}\left[\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\,z^{2j+3}-\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3}\,z^{2j+1}\right]\\
&= \frac{32}{\pi^3}\left[\sum_{j=1}^{\infty}\frac{1}{(2j-1)^3}\,z^{2j+1}-z-\sum_{j=1}^{\infty}\frac{1}{(2j+1)^3}\,z^{2j+1}\right]\\
&= \frac{32}{\pi^3}\left\{-z+\sum_{j=1}^{\infty}\left[\frac{1}{(2j-1)^3}-\frac{1}{(2j+1)^3}\right]z^{2j+1}\right\}\\
&= \frac{32}{\pi^3}\left[-z+\sum_{j=1}^{\infty}\frac{24j^2+2}{\left(4j^2-1\right)^3}\,z^{2j+1}\right]\\
&= \frac{32}{\pi^3}\,z\left[-1+\sum_{j=1}^{\infty}\frac{24j^2+2}{\left(4j^2-1\right)^3}\,z^{2j}\right].
\end{aligned}
$$

Unlike the original series, with coefficients that behave as $1/k^3$ (with $k = 2j+1$), this series has coefficients that go to zero as $1/k^4$ when $k \to \infty$, and therefore converges faster than the original one. This shows, in particular, that our evaluation of the set of dominant singularities of $w(z)$ was in fact correct. We have therefore for S_z the representation

$$
S_z = \frac{32}{\pi^3}\,\frac{z}{z^2-1}\left[-1+\sum_{j=1}^{\infty}\frac{24j^2+2}{\left(4j^2-1\right)^3}\,z^{2j}\right],
$$

with the singularities factored out. Although both this series and the original one are absolutely and uniformly convergent, this converges faster, and may be differentiated twice, still resulting in other series that are also absolutely and uniformly convergent.

We may now take the real and imaginary parts of the S_v series in order to obtain faster-converging representation of the original DP Fourier series and its FC series. We have on the unit circle, as we saw before in Subsection B.2,

$$
\frac{z}{z^2-1} = \frac{-\imath}{2\sin(\theta)},
$$

and therefore

$$
S_v = \frac{16}{\pi^3}\,\frac{-\imath}{\sin(\theta)}\left[-1+\sum_{j=1}^{\infty}\frac{24j^2+2}{\left(4j^2-1\right)^3}\,\cos(2j\theta)+\imath\sum_{j=1}^{\infty}\frac{24j^2+2}{\left(4j^2-1\right)^3}\,\sin(2j\theta)\right]
$$

$$
\begin{aligned}
= & \quad \frac{16}{\pi^3 \sin(\theta)} \left[\sum_{j=1}^{\infty} \frac{24j^2+2}{(4j^2-1)^3}\, \sin(2j\theta) \right] + \\
& +\imath\, \frac{16}{\pi^3 \sin(\theta)} \left[1 - \sum_{j=1}^{\infty} \frac{24j^2+2}{(4j^2-1)^3}\, \cos(2j\theta) \right].
\end{aligned}
$$

The original DP function is given by the imaginary part,

$$
f_{\rm s}(\theta) = \frac{16}{\pi^3 \sin(\theta)} \left[1 - \sum_{j=1}^{\infty} \frac{24j^2+2}{(4j^2-1)^3}\, \cos(2j\theta) \right],
$$

and the corresponding FC function $f_{\rm c}(\theta) = \bar{f}_{\rm s}(\theta)$ is given by the real part,

$$
f_{\rm c}(\theta) = \frac{16}{\pi^3 \sin(\theta)} \left[\sum_{j=1}^{\infty} \frac{24j^2+2}{(4j^2-1)^3}\, \sin(2j\theta) \right].
$$

Both of these series converge faster than the original Fourier series.

References

[1] J. L. deLyra, "Fourier Theory on the Complex Plane I – Conjugate Pairs of Fourier Series and Inner Analytic Functions", arXiv: 1409.2582.

[2] See, for example, the URL

`http://en.wikipedia.org/wiki/Abel's_theorem`

and the references therein.

[3] A paper reporting on the numerical tests of several simple examples of center series can be obtained at the URL

`http://latt.if.usp.br/scientific-pages/ntocs/`

A compressed tar file containing all the program used in that paper, and some associated utilities, can be found at that same URL.