Extension of the Schwarzschild Solution to the Case of Localized Fluid Matter I: Derivation and Properties

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Abstract

We work out, mostly but not completely by analytic methods, the solution of the Einstein gravitational field equation in the presence of a localized fluid-matter energy-momentum tensor, for a situation in which there is spherical symmetry, independence of the time, and a definite homogeneous equation of state. In this work we follow the arguments given, and use the results obtained and the conventions adopted in Dirac's marvelous little book on General Relativity [1]. We establish some of the main properties, and discuss in a general way some other properties, of the two-parameter class of solutions presented, including the nature of the geometry within the matter distribution and the possibility of the formation of event horizons.

1 Introduction

The purpose of this paper is to establish the general solution of the Einstein gravitational field equation, in the presence of localized sources, under a certain set of symmetry conditions. We will assume that we have static fluid matter with a spherically symmetric energy density and a definite equation of state, given by $P = \omega\rho$ where P is the pressure, ρ is the energy density, and ω is a positive real number in the interval $(0, 1/3]$. All the matter present will be assumed to be localized, that is, to be essentially all contained within a certain sphere whose surface is at the radial position r_S, for a sufficiently large value of r_S, satisfying the condition that $r_S \geq r_M$, where r_M is the Schwarzschild radius associated to the total mass M. In this paper we will use the terms "mass" and "energy" interchangeably, to refer to the total *energy* content of the fluid matter.

While the Schwarzschild solution is in fact a one-parameter family of solutions parametrized by the Schwarzschild radius r_M, what we will present here is a two-parameter family of solutions, parametrized by r_M and ω. The solution presented will be given mostly analytically, with the exception of a single dimensionless real function, which displays a fairly simple qualitative behavior, but which at least for now can only be obtained in detail numerically. However, we will see that the main properties of this function can be ascertained analytically.

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2 General Review

Here we will review the main points and equations relating to the Einstein field equations under the set of symmetry conditions that we are to impose on them. Our initial development here will follow closely the one presented in [1]. Under the conditions of time independence and of spherical symmetry around the origin of a spherical system of coordinates (t, r, θ, ϕ), the Schwarzschild system of coordinates, the most general possible metric is given by the invariant interval, written in terms of this spherical system of coordinates,

$$ds^2 = \mathrm{e}^{2\nu(r)}dt^2 - \mathrm{e}^{2\lambda(r)}dr^2 - r^2\left[d\theta^2 + \sin^2(\theta)d\phi^2\right], \tag{1}$$

where $\nu(r)$ and $\lambda(r)$ are two functions of only r. As one can see, in this work we will use the time-like signature $(+, -, -, -)$, following [1]. Expressing the coefficients of dt^2 and dr^2 by the exponentials shown ensures that they have the physically required signs.

Let us now comment on the physical interpretation of these coordinates. In this coordinate system r is such that a sphere centered at the origin $r = 0$ and with its surface located at the position r has total physical surface area equal to $4\pi r^2$. However, r is *not* the true physical distance from the surface of the sphere to the origin, and furthermore dr is *not* a variation of physical length in the radial direction. Note also that the time variable t is *not* the true proper time at each spatial position. These two coordinates only recover their usual meanings in the $r \to \infty$ asymptotic limit, if that limit is accessible. In general the element of physical length in the radial direction is given by $\exp[\lambda(r)]dr$, and the element of proper time at each position is given by $\exp[\nu(r)]dt$. Finally note that, on the other hand, the quantities $rd\theta$ and $r\sin(\theta)d\phi$ are in fact true physical elements of arc length on the surface of the sphere, that is, on the spherical surface located at the position r.

From this invariant interval we can simply read out the metric tensor $g_{\mu\nu}$, that is, the metric tensor in its covariant form,

$$g_{\mu\nu} = \begin{bmatrix} \mathrm{e}^{2\nu(r)} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{2\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta) \end{bmatrix}. \tag{2}$$

Note that we have for the determinant $g = \det(g_{\mu\nu})$ of the matrix $g_{\mu\nu}$ the value

$$\sqrt{-g} = \mathrm{e}^{\nu(r)+\lambda(r)}\, r^2 \sin(\theta), \tag{3}$$

which is not zero for $r > 0$ except at the two poles $\theta = 0$ and $\theta = \pi$, which are just the usual singularities of the spherical system of coordinates, and can therefore be safely ignored. Recalling that $g_\mu^\nu = \delta_\mu^\nu$ is the identity matrix, we obtain at once the metric tensor $g^{\mu\nu}$, that is, the metric tensor in its contravariant form, which is the inverse matrix to the diagonal matrix $g_{\mu\nu}$ given above. The matrix $g_{\mu\nu}$ in invertible almost everywhere due to the fact that $g \neq 0$ almost everywhere, and the inverse is immediately found to be given by

$$g^{\mu\nu} = \begin{bmatrix} \mathrm{e}^{-2\nu(r)} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}(\theta) \end{bmatrix}. \tag{4}$$

The next step in the geometric development is to calculate from $g_{\mu\nu}$ and $g^{\mu\nu}$ the Christoffel symbol of the second kind $\Gamma^\alpha{}_{\mu\nu}$, which gives us the metric-compatible and torsion-free

$$\Gamma^{0}{}_{\mu\nu} = \begin{bmatrix} 0 & \nu'(r) & 0 & 0 \\ \nu'(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma^{1}{}_{\mu\nu} = \begin{bmatrix} \nu'(r)\,\mathrm{e}^{2\nu(r)-2\lambda(r)} & 0 & 0 & 0 \\ 0 & \lambda'(r) & 0 & 0 \\ 0 & 0 & -r\,\mathrm{e}^{-2\lambda(r)} & 0 \\ 0 & 0 & 0 & -r\sin^2(\theta)\,\mathrm{e}^{-2\lambda(r)} \end{bmatrix},$$

$$\Gamma^{2}{}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta)\cos(\theta) \end{bmatrix},$$

$$\Gamma^{3}{}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^{-1} \\ 0 & 0 & 0 & \cot(\theta) \\ 0 & r^{-1} & \cot(\theta) & 0 \end{bmatrix}.$$

Table 1: The values of the components of the connection $\Gamma^{\alpha}{}_{\mu\nu}$.

connection for the pseudo-Riemannian parallel transport in spacetime. It is given in terms of the metric tensor by

$$\begin{aligned} \Gamma^{\alpha}{}_{\mu\nu} &= g^{\alpha\sigma}\Gamma_{\sigma\mu\nu} \\ &= g^{\alpha\sigma}\frac{1}{2}\left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right). \end{aligned} \tag{5}$$

This is the Christoffel symbol of the second kind, while the same quantity with all indices downstairs is the Christoffel symbol of the first kind. They are symmetric on the last two indices, as one can see, and they are non-tensors, because the derivative of a second-rank tensor is *not* a tensor. Note that the three terms within parenthesis correspond to cyclic permutations of the three indices. From now on the derivatives with respect to r of $\nu(r)$, $\lambda(r)$, and any other quantities that depend only on r, will be denoted by primes.

In this calculation many of the components of $\Gamma^{\alpha}{}_{\mu\nu}$ turn out to be zero, as reported in [1], and if one recalls that this quantity is symmetric on the pair of indices (μ, ν), one gets the results, written in matrix form on the two lower indices, that are shown in Table 1. This matrix form is very useful as a basis for further calculations, such as that of the curvature tensor. Using these expressions in the equation defining the Ricci curvature tensor $R_{\mu\nu}$ in terms of the connection, which is given by

$$R_{\mu\nu} = \partial_\nu\Gamma^{\alpha}{}_{\mu\alpha} - \partial_\alpha\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha}, \tag{6}$$

one gets a diagonal matrix for this curvature tensor, in its covariant form,

$$R_{\mu\nu} = \left[\begin{array}{llll} R_{00} & 0 & 0 & 0 \\ 0 & R_{11} & 0 & 0 \\ 0 & 0 & R_{22} & 0 \\ 0 & 0 & 0 & R_{33} \end{array}\right], \tag{7}$$

with the four diagonal elements given in [1], which are

$$\begin{aligned} R_{00} &= \left\{-\nu''(r) + \lambda'(r)\nu'(r) - \left[\nu'(r)\right]^2 - \frac{2\nu'(r)}{r}\right\} e^{2\nu(r)-2\lambda(r)}, \\ R_{11} &= \left\{\nu''(r) - \lambda'(r)\nu'(r) + \left[\nu'(r)\right]^2 - \frac{2\lambda'(r)}{r}\right\}, \\ R_{22} &= \left\{\left[1 + r\nu'(r) - r\lambda'(r)\right] e^{-2\lambda(r)} - 1\right\}, \\ R_{33} &= \left\{\left[1 + r\nu'(r) - r\lambda'(r)\right] e^{-2\lambda(r)} - 1\right\} \sin^2(\theta). \end{aligned} \tag{8}$$

With the use of $g^{\mu\nu}$ the same quantities can be written, in mixed form, as

$$\begin{aligned} R_0^0 &= \left\{-\nu''(r) + \lambda'(r)\nu'(r) - \left[\nu'(r)\right]^2 - \frac{2\nu'(r)}{r}\right\} e^{-2\lambda(r)}, \\ R_1^1 &= \left\{-\nu''(r) + \lambda'(r)\nu'(r) - \left[\nu'(r)\right]^2 + \frac{2\lambda'(r)}{r}\right\} e^{-2\lambda(r)}, \\ R_2^2 &= -\left[\frac{1}{r^2} + \frac{\nu'(r)}{r} - \frac{\lambda'(r)}{r}\right] e^{-2\lambda(r)} + \frac{1}{r^2}, \\ R_3^3 &= -\left[\frac{1}{r^2} + \frac{\nu'(r)}{r} - \frac{\lambda'(r)}{r}\right] e^{-2\lambda(r)} + \frac{1}{r^2}. \end{aligned} \tag{9}$$

We will tend to write all relevant tensor quantities in this mixed form, which is the most useful for our purposes here. Note that the exponential $\exp[2\nu(r)]$ has vanished from these expressions, which therefore contain only the functions $\lambda(r)$, $\lambda'(r)$, $\nu'(r)$ and $\nu''(r)$. Note also that it turns out that $R_2^2 = R_3^3$, as a consequence of the symmetries that we imposed. The last geometric element that we need to discuss here is the scalar curvature $R = R_\mu^\mu$, which can be written as

$$R = R_0^0 + R_1^1 + R_2^2 + R_3^3. \tag{10}$$

We therefore can write our result for the scalar curvature as

$$\frac{1}{2}\,R = \left\{-\nu''(r) + \lambda'(r)\nu'(r) - \left[\nu'(r)\right]^2 - \frac{2\nu'(r)}{r} + \frac{2\lambda'(r)}{r} - \frac{1}{r^2}\right\} e^{-2\lambda(r)} + \frac{1}{r^2}. \tag{11}$$

In the theory of General Relativity the equation determining the gravitational field is written in terms of the Ricci curvature tensor $R_{\mu\nu}$. The equation also involves the matter energy-momentum tensor $T_{\mu\nu}$, which plays the role of the source of the gravitational field. The Einstein gravitational field equation is a tensor equation which, written in its mixed form, using our notation here, with the signature $(+,-,-,-)$, following [1], is given by

$$R_\mu^\nu - \frac{1}{2}\,R\,g_\mu^\nu = -\kappa\,T_\mu^\nu, \tag{12}$$

where $\kappa = 8\pi G/c^4$, G is the universal gravitational constant and c is the speed of light.

Note that the imposition of spherical symmetry and time independence on the solution of the Einstein field equation reduces the problem of finding that solution to a much simpler one-dimensional one, on the variable r. While one can take any metric at all, so long as the functions involved in it, such as $\nu(r)$ and $\lambda(r)$, are differentiable to the second order, and just calculate R_{μ}^{ν} and R in order to simply verify whatever results for T_{μ}^{ν}, with such a deeply non-linear equation one is *not* free to choose the matter energy-momentum tensor T_{μ}^{ν} in an arbitrary way. Both the general structure of the theory and the symmetry conditions will impose restrictions on the possible values of this tensor. For example, since we have, due to the imposition of the symmetries, that $R_2^2 = R_3^3$, and since we also have that $g_2^2 = g_3^3$, it follows at once that $T_2^2 = T_3^3$. Also, since R_{μ}^{ν} and g_{μ}^{ν} are symmetric tensors, so must be T_{μ}^{ν}.

At this point we must pause in order to consider what information we have obtained so far about T_{μ}^{ν}. First of all, since under the current hypotheses the left-hand side of the Einstein field equation turns out to be *diagonal*, and since we have also the additional fact that the expressions of the last two component equations turn out to be identical, the same must be true for the matter energy-momentum tensor T_{μ}^{ν} on the right-hand side of the equation, which must therefore be diagonal,

$$T_{\mu}^{\nu} = \begin{bmatrix} T_0^0(r) & 0 & 0 & 0 \\ 0 & T_1^1(r) & 0 & 0 \\ 0 & 0 & T_2^2(r) & 0 \\ 0 & 0 & 0 & T_3^3(r) \end{bmatrix}, \tag{13}$$

and which must also satisfy $T_2^2(r) = T_3^3(r)$. In addition to this, since there are no dependencies on t, θ or ϕ, it follows that T_{μ}^{ν} can depend only on r. Note, however, that we still do *not* have any further information about any possible relations between $T_0^0(r)$, $T_1^1(r)$ and $T_2^2(r)$. From this point on, in order to simplify the notation, we will use the variable names $T_0(r)$, $T_1(r)$, $T_2(r)$ and $T_3(r)$ for the diagonal elements $T_0^0(r)$, $T_1^1(r)$, $T_2^2(r)$ and $T_3^3(r)$, respectively, of the energy-momentum tensor T_{μ}^{ν} in its mixed form.

The main general consistency condition imposed by the structure of the theory is that the covariant divergence of T_{μ}^{ν} must vanish, that is, the condition that we must have that

$$D_{\nu} T_{\mu}^{\nu} = 0. \tag{14}$$

This is due to the fact that the combination of tensors that constitutes the left-hand side of the Einstein field equation satisfies the Bianci identity of the Ricci curvature tensor,

$$D_{\nu}\left(R^{\mu\nu} - \frac{1}{2}\, R\, g^{\mu\nu}\right) = 0, \tag{15}$$

which therefore implies the requirement that the covariant divergence of $T^{\mu\nu}$ must vanish,

$$D_{\nu} T^{\mu\nu} = 0. \tag{16}$$

Since $g_{\mu\nu}$ and $g^{\mu\nu}$ behave as constants under covariant differentiation, we may then write this condition as the requirement on T_{μ}^{ν} given in Equation (14). At this point we must calculate this consistency condition for the specific case of time independence and spherical symmetry. In other words, we must now write explicitly, under these conditions, the expression shown in Equation (14). This rather long calculation is done in detail in Section A.1 of Appendix A. As one can see there, three of the four conditions in Equation (14) are automatically satisfied, since it results from that calculation that

$$
\begin{aligned}
D_\nu T_0^\nu(r) &\equiv 0,\\
D_\nu T_2^\nu(r) &\equiv 0,\\
D_\nu T_3^\nu(r) &\equiv 0.
\end{aligned} \tag{17}
$$

Therefore, the only non-trivial condition is that given by $D_\nu T_1^\nu(r) = 0$, which results in

$$
\left[r\nu'(r)\right][T_0(r) - T_1(r)] = \left[rT_1'(r)\right] + [2T_1(r) - T_2(r) - T_3(r)], \tag{18}
$$

and which can also be written as

$$
\left[r\nu'(r)\right] = \frac{[rT_1'(r)]}{T_0(r) - T_1(r)} + \frac{2T_1(r) - T_2(r) - T_3(r)}{T_0(r) - T_1(r)}. \tag{19}
$$

Note that this consistency condition on $T_\mu^\nu(r)$ ends up involving only the element of the metric given by the function $\nu'(r)$.

We now have at hand all the elements needed in order to write the Einstein gravitational field equation under our hypotheses about the geometry, as well as the relevant consistency condition. Using the elements R_μ^ν and R, as well as the fact that $g_\mu^\nu = \delta_\mu^\nu$, we may now write the left-hand side of the components of the Einstein field equation, in mixed form, as

$$
\begin{aligned}
R_0^0 - \frac{1}{2}R\,g_0^0 &= \left[\frac{1}{r^2} - \frac{2\lambda'(r)}{r}\right] \mathrm{e}^{-2\lambda(r)} - \frac{1}{r^2},\\
R_1^1 - \frac{1}{2}R\,g_1^1 &= \left[\frac{1}{r^2} + \frac{2\nu'(r)}{r}\right] \mathrm{e}^{-2\lambda(r)} - \frac{1}{r^2},\\
R_2^2 - \frac{1}{2}R\,g_2^2 &= -\left\{-\nu''(r) + \lambda'(r)\nu'(r) - \left[\nu'(r)\right]^2 - \frac{\nu'(r)}{r} + \frac{\lambda'(r)}{r}\right\} \mathrm{e}^{-2\lambda(r)},\\
R_3^3 - \frac{1}{2}R\,g_3^3 &= -\left\{-\nu''(r) + \lambda'(r)\nu'(r) - \left[\nu'(r)\right]^2 - \frac{\nu'(r)}{r} + \frac{\lambda'(r)}{r}\right\} \mathrm{e}^{-2\lambda(r)}.
\end{aligned} \tag{20}
$$

It thus results that we get some fairly simple expressions for the left-hand sides of the four components of the field equation in mixed form. Note once more that the exponential $\exp[2\nu(r)]$ is absent from these expressions, and also that the expressions of the last two component equations are identical, and therefore not independent from each other. We must now write the right-hand side of these equations, and thus introduce the energy-momentum tensor $T_\mu^\nu(r)$ that was discussed before.

Let us recall that, for an homogeneous and isotropic *cosmological* spacetime, containing equally homogeneous and isotropic fluid matter, in the co-moving system of coordinates, in which the matter is locally at rest, we would have that $T_0(r) = \rho(r)$, where $\rho(r)$ is the energy density, and that $T_1(r) = T_2(r) = T_3(r) = -P(r)$, where $P(r)$ is the pressure. We can expect a similar situation in our case here, for the values of the components of the energy-momentum tensor. Note that the *pure radiation* condition given by the equation of state $\rho(r) = 3P(r)$ translates here to the simple *invariant* condition $T(r) = 0$ on the trace of $T_\mu^\nu(r)$. Note also that this condition holds for any type of massless matter field. It is important to observe that up to this point we have assumed no more about these quantities than what is implied by the structure of the field equation itself. Since we must have that $T_2(r) = T_3(r)$ as a consequence of the symmetries that we imposed, we have only three independent components of the field equation, to which we now add the consistency condition as an ancillary condition,

$$
\left[\frac{1}{r^2} - \frac{2\lambda'(r)}{r}\right] \mathrm{e}^{-2\lambda(r)} - \frac{1}{r^2} = -\kappa\,T_0(r),
$$

$$
\begin{aligned}
\left[\frac{1}{r^2}+\frac{2\nu'(r)}{r}\right]\mathrm{e}^{-2\lambda(r)}-\frac{1}{r^2} &= -\kappa\,T_1(r),\\
\Big\{\nu''(r)-\lambda'(r)\nu'(r)+ & \\
+\left[\nu'(r)\right]^2+\frac{\nu'(r)}{r}-\frac{\lambda'(r)}{r}\Big\}\,\mathrm{e}^{-2\lambda(r)} &= -\kappa\,T_2(r),\\
\left[r\nu'(r)\right]=\frac{\left[rT_1'(r)\right]}{T_0(r)-T_1(r)}+\frac{2T_1(r)-T_2(r)-T_3(r)}{T_0(r)-T_1(r)}, & \qquad (21)
\end{aligned}
$$

where we recall that $T_3(r)=T_2(r)$. We will now impose on the components of $T_\mu^{\nu}(r)$ the equation of state for fluid matter. Since the equation of state determines the nature of the fluid, and assuming that no phase transitions occur within the volume occupied by the matter, we must have the same equation of state throughout the volume of the fluid matter. In other words, the equation of state must not dependent on the position r. Both the energy density $\rho(r)$ and the pressure $P(r)$ may depend on r, but the relations between them may not. This means that we are assuming, in this simplest case, that there is a certain homogeneity regarding the *state* of the matter, which is assumed not to undergo a phase transition along the possible values of r. This implies that we should have the relations

$$
\begin{aligned}
T_1(r) &= -\omega T_0(r),\\
T_2(r) &= -\omega T_0(r),\\
T_3(r) &= -\omega T_0(r), \qquad (22)
\end{aligned}
$$

which automatically satisfy the condition that $T_3(r)=T_2(r)$, and where ω is a positive real number in the interval $(0,1/3]$. The value $\omega=0$ corresponds to pressureless dust and the value $\omega=1/3$ corresponds to pure relativistic radiation. The value of ω in this range determines what fraction of the energy is bound in the form or rest mass and what fraction is in the form of relativistic radiation. Multiplying the first three component equations in Equation (21) by r^2 and making the replacements indicated above, which result in the fluid matter being described by the single function $T_0(r)$, we get

$$
\begin{aligned}
\left\{1-2\left[r\lambda'(r)\right]\right\}\mathrm{e}^{-2\lambda(r)} &= 1-\kappa r^2T_0(r),\\
\left\{1+2\left[r\nu'(r)\right]\right\}\mathrm{e}^{-2\lambda(r)} &= 1+\omega\kappa r^2T_0(r),\\
\Big\{r^2\nu''(r)-\left[r\lambda'(r)\right]\left[r\nu'(r)\right]+ & \\
+\left[r\nu'(r)\right]^2+\left[r\nu'(r)\right]-\left[r\lambda'(r)\right]\Big\}\,\mathrm{e}^{-2\lambda(r)} &= \omega\kappa r^2T_0(r),\\
\left[r\nu'(r)\right]=-\frac{\omega}{1+\omega}\,\frac{\left[rT_0'(r)\right]}{T_0(r)}. & \qquad (23)
\end{aligned}
$$

At this point it is convenient to write all the equations in terms of the single function given by $\mathfrak{T}(r)=\kappa r^2T_0(r)$, which also happens to be dimensionless, because due to the dimensions of the Einstein field equation the quantity $\kappa T_0(r)$ has the dimensions of $[\text{length}]^{-2}$,

$$
\left\{1-\left[2r\lambda'(r)\right]\right\}\mathrm{e}^{-2\lambda(r)} = 1-\mathfrak{T}(r), \qquad (24)
$$

$$
\left\{1+\left[2r\nu'(r)\right]\right\}\mathrm{e}^{-2\lambda(r)} = 1+\omega\mathfrak{T}(r), \qquad (25)
$$

$$
\Big\{r^2\nu''(r)-\left[r\lambda'(r)\right]\left[r\nu'(r)\right]+
$$

$$+\left[r\nu'(r)\right]^2+\left[r\nu'(r)\right]-\left[r\lambda'(r)\right]\Big\}\,\mathrm{e}^{-2\lambda(r)} = \omega\mathfrak{T}(r), \tag{26}$$

$$\left[r\nu'(r)\right]=\frac{2\omega}{1+\omega}-\frac{\omega}{1+\omega}\,\frac{\left[r\mathfrak{T}'(r)\right]}{\mathfrak{T}(r)}. \tag{27}$$

These are the four equations that must be satisfied by the functions $\nu(r)$, $\lambda(r)$ and $\mathfrak{T}(r)$ that represent a solution of the Einstein gravitational field equation in the presence of fluid matter, under the hypotheses of time independence, spherical symmetry and a simple homogeneous local equation of state for the fluid matter. In the sequence we will first recover the solution for empty space, that is, we will derive from these equations the Schwarzschild solution, and then we will establish the solution in the presence of the fluid matter. Ultimately, the Schwarzschild solution will play the role of being the $r\to\infty$ asymptotic limit of the solution in the presence of localized fluid matter.

3 Solution in Vacuum

The Schwarzschild solution corresponds to the vacuum case, in which there in no matter present in the region where we are to determine the metric. For us here, this will be the solution outside the sphere that contains essentially all the matter, a region, it should be noted, that is continuously connected to radial infinity. In this empty-space case, which therefore corresponds to $T_\mu^\nu=0$, the t and r component equations given in Equations (24) and (25) reduce to

$$\begin{aligned}\left\{1-2\left[r\lambda'(r)\right]\right\}\,\mathrm{e}^{-2\lambda(r)} &= 1,\\ \left\{1+2\left[r\nu'(r)\right]\right\}\,\mathrm{e}^{-2\lambda(r)} &= 1,\end{aligned} \tag{28}$$

so that subtracting the two equations we have that

$$\begin{aligned}\left\{-2\left[r\lambda'(r)\right]-2\left[r\nu'(r)\right]\right\}\,\mathrm{e}^{-2\lambda(r)} &= 0 \quad\Rightarrow\\ \lambda'(r)+\nu'(r) &= 0,\end{aligned} \tag{29}$$

assuming that r and the exponential $\exp[-2\lambda(r)]$ are never zero within the region of interest. This implies that $\lambda(r)+\nu(r)$ must be a constant C, and since in the $r\to\infty$ asymptotic limit both $\lambda(r)$ and $\nu(r)$ must go to zero in order for spacetime to approach the usual flat Lorentzian spacetime, it follows that the constant must be $C=0$. It therefore follows that $\nu(r)=-\lambda(r)$, and hence that we are left with the single equation for $\lambda(r)$ given by

$$\begin{aligned}\left[1-2r\lambda'(r)\right]\,\mathrm{e}^{-2\lambda(r)} &= 1 \quad\Rightarrow\\ \left[r\,\mathrm{e}^{-2\lambda(r)}\right]' &= 1 \quad\Rightarrow\\ r\,\mathrm{e}^{-2\lambda(r)}-r_1\,\mathrm{e}^{-2\lambda(r_1)} &= r-r_1,\end{aligned} \tag{30}$$

where we integrated from some arbitrary reference point r_1 to r. We have therefore that

$$\mathrm{e}^{-2\lambda(r)}=1-\frac{r_1}{r}\left[1-\mathrm{e}^{-2\lambda(r_1)}\right]. \tag{31}$$

If we recall that $\exp[2\lambda(r)]$ is the coefficient of dr^2 in the invariant interval, we realize that there will be a singularity in the coordinate system if and when $\exp[-2\lambda(r)]=0$, and that therefore we can take the value of the arbitrary reference point r_1 down only to the point r_M at which this condition holds. If we make $r_1=r_M$, we have that

$$\begin{aligned} e^{-2\lambda(r_1)} &= e^{-2\lambda(r_M)} \\ &= 0, \end{aligned} \tag{32}$$

and therefore our solution for $\lambda(r)$, for $r > r_M$, becomes

$$e^{-2\lambda(r)} = 1 - \frac{r_M}{r}, \tag{33}$$

where for $r \to r_M$ we have that $\lambda(r) \to \infty$. One can see that this expression has indeed the expected behavior of vanishing at the point $r = r_M$. Furthermore, since $\nu(r) = -\lambda(r)$, we also have that

$$e^{2\nu(r)} = 1 - \frac{r_M}{r}, \tag{34}$$

where for $r \to r_M$ we have that $\nu(r) \to -\infty$. This completes the determination of the metric, except for the value of r_M. Note that when one solves the gravitational field equation in this fashion one loses quite completely the explicit local connection with the sources. In a way, one of the main objectives of this paper is to recover that direct connection. In order to recover this connection in an *indirect* way, one takes recourse to a comparison with the Newtonian limit for large values of r, which implies that $r_M = 2MG/c^2$, where M is the total mass of the central source, as shown in [1]. The solution is therefore valid only outside the sources, and in a region which must be continuously connected to the $r \to \infty$ asymptotic limit. The complete metric is then given by

$$ds^2 = \left(1 - \frac{r_M}{r}\right) dt^2 - \left(1 - \frac{r_M}{r}\right)^{-1} dr^2 - r^2 \left[d\theta^2 + \sin^2(\theta) d\phi^2\right], \tag{35}$$

for $r > r_M$. This is indeed the well-known Schwarzschild metric. We must now verify that this solution in fact satisfies also the θ component equation and the consistency equation shown in Equations (26) and (27). That this last one is satisfied is obvious since all the components $T_\mu(r)$ are identically zero and therefore $\mathfrak{T}(r) \equiv 0$. In order to see this we may simply write that equation in the form

$$\mathfrak{T}(r) \left[r\nu'(r)\right] = \frac{2\omega}{1+\omega} \mathfrak{T}(r) - \frac{\omega}{1+\omega} r\mathfrak{T}'(r), \tag{36}$$

which makes this fact quite plain. In order to be able to verify Equation (26) we must first calculate the relevant derivatives of the functions

$$\begin{aligned} \nu(r) &= \frac{1}{2} \ln\left(\frac{r - r_M}{r}\right), \\ \lambda(r) &= \frac{1}{2} \ln\left(\frac{r}{r - r_M}\right), \end{aligned} \tag{37}$$

that characterize the metric. Calculating the first derivatives of these two quantities, as well as the second derivative of $\nu(r)$, we have for the three relevant derivatives

$$\begin{aligned} r\nu'(r) &= \frac{1}{2} \frac{r_M}{r - r_M}, \\ r\lambda'(r) &= -\frac{1}{2} \frac{r_M}{r - r_M}, \\ r^2\nu''(r) &= \frac{1}{2} \frac{r_M^2 - 2rr_M}{(r - r_M)^2}. \end{aligned} \tag{38}$$

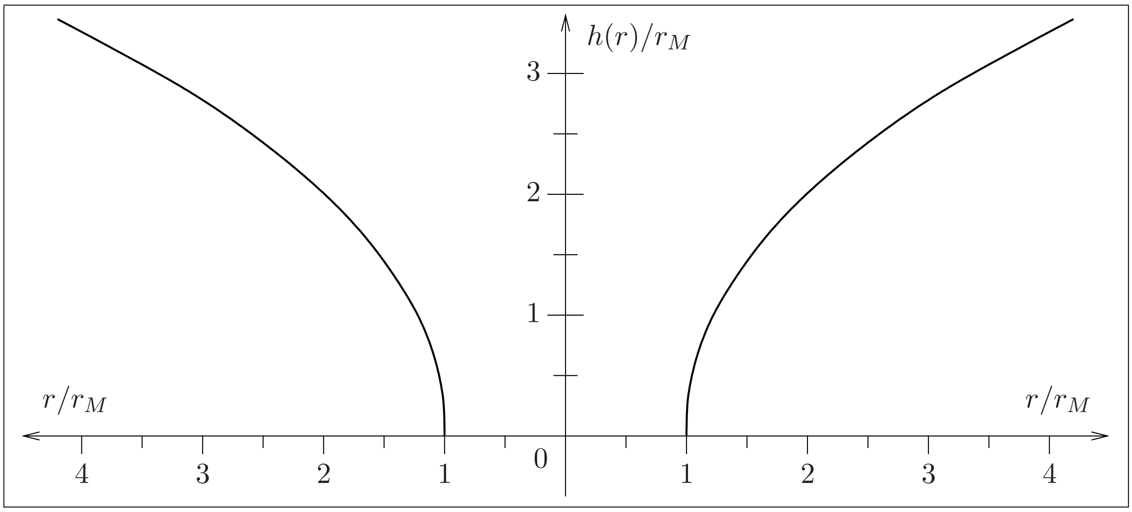

Figure 1: The embedding in a flat three-dimensional space of a two-dimensional spatial section through the origin of the Schwarzschild solution.

Using the facts that $\lambda'(r) = -\nu'(r)$ and that $\mathfrak{T}(r) = 0$ the θ component equation shown in Equation (26) can be written as

$$\begin{aligned} \left\{ r^2\nu''(r) + 2\left[r\nu'(r)\right]^2 + 2\left[r\nu'(r)\right] \right\} \mathrm{e}^{-2\lambda(r)} &= 0 \quad \Rightarrow \\ r^2\nu''(r) + 2\left[r\nu'(r)\right]^2 + 2\left[r\nu'(r)\right] &= 0, \end{aligned} \tag{39}$$

since we have that $\exp[-2\lambda(r)] \neq 0$ for $r > r_M$. Substituting the values of the derivatives in the expression on the left-hand side of this equation we have that

$$\begin{aligned} &\frac{1}{2}\,\frac{r_M^2 - 2rr_M}{(r-r_M)^2} + \frac{2}{4}\,\frac{r_M^2}{(r-r_M)^2} + \frac{2}{2}\,\frac{r_M}{r-r_M} \\ &\quad = \frac{1}{2}\,\frac{r_M^2 - 2rr_M + r_M^2 + 2rr_M - 2r_M^2}{(r-r_M)^2} \\ &\quad = 0, \end{aligned} \tag{40}$$

which shows that this equation is in fact satisfied. Therefore, we have now shown that all the relevant equations are satisfied by this vacuum solution, for $r > r_M$.

While from the point of view of the differential geometry by itself there is no singularity at $r = r_M$, so that it is quite possible to extend this solution to the region where $r < r_M$, it is not possible to do so using the Schwarzschild coordinate system. It is necessary to change to other systems of coordinates, and in doing so one loses the physical interpretations associated to the Schwarzschild coordinates. The basic problem is that, as one can immediately see in the Schwarzschild solution, when one crosses the r_M boundary the physical roles of the radial and temporal coordinates get interchanged, due to the changes in sign of the factors multiplying dt^2 and dr^2 in the invariant interval. The interpretational issues arising from this will not be discussed here, since they are irrelevant for our current purposes. All that matters to us is the Schwarzschild solution for $r > r_M$, which will be used as a guide in the construction of the extended solution, valid in the presence of the localized fluid matter, and which will become the $r \to \infty$ asymptotic limit of that solution.

We may build a visualization of a two-dimensional spatial section of this vacuum solution through the origin by means of a isometric embedding of the two-dimensional spatial section in a three-dimensional flat space, a section of which is shown in Figure 1. In order to do this we fix t at zero, θ at $\pi/2$, and let r and ϕ vary so as to span a plane. The two-dimensional spatial interval of the resulting two-dimensional section is given by

$$d\ell^2 = \frac{r}{r - r_M}\, dr^2 + r^2 d\phi^2. \tag{41}$$

We then introduce an embedding variable $h(r)$ such that in the flat three-dimensional embedding space spanned by (r, ϕ, h) we have for the element of length $d\lambda$

$$d\lambda^2 = dr^2 + r^2 d\phi^2 + dh^2, \tag{42}$$

using the cylindrical system of coordinates (r, ϕ, h). If we fix ϕ at some arbitrary value, making $d\phi = 0$, and impose that the length element $d\lambda$ is given by the physical length of the Schwarzschild solution associated to a variation dr,

$$d\lambda^2 = \frac{r}{r - r_M}\, dr^2, \tag{43}$$

then in the (r, h) plane of the embedding space, which is shown in Figure 1, we have that

$$\begin{aligned} dr^2 + dh^2 &= \frac{r}{r - r_M}\, dr^2 \quad \Rightarrow \\ dh^2 &= \frac{r_M}{r - r_M}\, dr^2. \end{aligned} \tag{44}$$

This relation between dh and dr can be integrated to yield a function $h(r)$, and therefore a two-dimensional curved surface within the three-dimensional flat embedding space, which is described by the variables r and ϕ. The metric geometry over such a surface is that given by the Schwarzschild solution for the section through the origin, and therefore this is an isometric embedding of that two-dimensional geometry. Doing the integration we have

$$\begin{aligned} dh &= \frac{\sqrt{r_M}}{\sqrt{r - r_M}}\, dr \quad \Rightarrow \\ h(r) &= 2\sqrt{r_M}\sqrt{r - r_M}, \end{aligned} \tag{45}$$

where we integrated on r, choosing the integration constant in such a way that $h(r_M) = 0$. If we invert this relation a parabola results, giving r in terms of h,

$$r(h) = r_M + \frac{h^2}{4r_M}, \tag{46}$$

for all ϕ. If we rotate the tilted parabola shown on the right-hand side of Figure 1 around the vertical h axis, that is, if we now let ϕ vary from 0 to 2π, this simple embedding illustrates the two-dimensional form of the mouth of the famous “wormhole”, immediately outside the event horizon located at $r = r_M$, that is, for $r \geq r_M$ and $h \geq 0$.

For future comparison with the solution in the presence of matter we record here the values of the following quantities in the case of the Schwarzschild solution. This vacuum solution is characterized by the set of quantities

$$e^{2\nu(r)} = \frac{r - r_M}{r}, \tag{47}$$

$$e^{2\lambda(r)} = \frac{r}{r - r_M}, \tag{48}$$

$$r\nu'(r) = \frac{1}{2}\frac{r_M}{r - r_M}, \tag{49}$$

$$r\lambda'(r) = -\frac{1}{2}\frac{r_M}{r - r_M}, \tag{50}$$

$$r\left[r\nu'(r)\right]'(r) = -\frac{1}{2}\frac{r r_M}{(r - r_M)^2}, \tag{51}$$

where we used, in order to get from Equation (38) to Equation (51), the fact that

$$r\left[r\nu'(r)\right]' = r^2\nu''(r) + r\nu'(r). \tag{52}$$

These quantities are to be interpreted as the asymptotic values of the corresponding quantities for the solution in the presence of fluid matter. This completes the discussion of the Schwarzschild solution, and therefore we proceed now to the discussion of the solution in the presence of localized fluid matter.

4 Solution with Fluid Matter

We will now describe a method for obtaining the solution in the presence of fluid matter. We start with an informed guess about the form of the function $\lambda(r)$. The ansatz that we will present here is suggested by the well-known Jebsen-Birkhoff [2, 3] theorem, which states that any spherically symmetric solution of the vacuum field equation must be both static and asymptotically flat, and therefore must be given by the Schwarzschild metric. It is to be noted, however, that this very statement implies, as a matter of course, that the solution at issue lies in a region that has continuous access to radial infinity.

It is usually stated that the theorem also implies that the geometry within the vacuous region between two spherically symmetric concentric shells, which do not need to be thin, is given by a radial section of the Schwarzschild solution, with the corresponding internal mass, between the corresponding two radii. However, this is not entirely correct, as one can see in the discussion presented in [4]. While it is correct for the *spatial* part of the geometry, there is a change in the *temporal* part. This can be seen in simple terms if one realizes that there must be a red-shift/blue-shift relationship between the internal bounded vacuous region and the external region at radial infinity. In order to see this it suffices to consider a monochromatic beam of light propagating from radial infinity towards the localized matter distribution, and passing through a thin radial hole made across the outer shell, into the bounded vacuous region. It is quite clear that the blue shift undergone by this beam of light is *not* the same that one would get in the absence of the outer shell, since it includes the blue-shift effect of the mass in the outer shell. Therefore, the coefficient $\exp[2\nu(r)]$ of the term dt^2 of the invariant interval, which gives this blue shift, must differ from the one in the Schwarzschild solution for the internal mass.

In short, we may safely assume that the Jebsen-Birkhoff theorem does have the interesting consequence that the geometry within a spherically symmetric empty shell of mass must be given by a flat Minkowski metric. In other words, spacetime is flat there, and thus the gravitational field must vanish inside an empty spherically symmetric shell, just as is the case for Newtonian gravitation. However, there is a difference between this bounded flat region and the flat space at radial infinity, because the relative rates of the passage of time differ between the two regions. In other words, while one should expect that, in the bounded vacuous region between two concentric shells, the metric should be such that the radial factor given by $\exp[2\lambda(r)]$ would be the one in the Schwarzschild solution, with the total mass that exists strictly within the external shell, one should *not* expect the same to be true for the temporal factor given by $\exp[2\nu(r)]$, which should display some significant difference with respect to the corresponding factor of the Schwarzschild solution for the internal mass. In fact, it is not difficult to see that the imposition of the condition $\nu(r) = -\lambda(r)$ on the component equations given in Equations (24) and (25) implies at once that $\mathfrak{T}(r) \equiv 0$, thus leading us back to the vacuum solution.

These facts strongly suggest that, in the case of the problem in the presence of fluid matter, which we are considering in this paper, the spatial part of the geometry at the position r is that due only to the mass within the sphere whose surface is at the position r, and that at that location the solution should be given by the spatial part of the Schwarzschild metric with an appropriate value of the mass. If we consider the factor $\exp[2\lambda(r)]$ in the dr^2 term of the invariant interval, and its value in the case of the Schwarzschild solution, we are immediately led to consider writing this factor in the following way for the case of the solution in the presence of fluid matter,

$$e^{2\lambda(r)} = \frac{r}{r - r_M\beta(r)}, \tag{53}$$

where $r_M = 2MG/c^2$ is the Schwarzschild radius associated to the *total* mass M of the distribution of matter, and $\beta(r)$ is a dimensionless function, *presumably* with values in the interval $[0,1]$, so that $r_M\beta(r)$ effectively corresponds to a certain fraction of that total mass. This is the ansatz that we will use here. The case $\beta(r) \equiv 1$ corresponds, of course, to the value of the quantity $\exp[2\lambda(r)]$ for the case of the original Schwarzschild solution. Therefore, it is to be expected that in the $r \to \infty$ asymptotic limit we will get $\beta(r) \to 1$. Of course, the correctness of this ansatz will have to be tested by the successful imposition of the field equation, which is what we will go on to do right away.

Therefore, let us consider each component equation in turn and thus obtain expressions for all the relevant quantities in terms of the single function $\beta(r)$. Starting from this ansatz, we may now use the t component of the field equation, shown in Equation (24), in order to get the dimensionless quantity $\mathfrak{T}(r)$. That equation can be written in the form

$$\begin{aligned} \left[r\, e^{-2\lambda(r)}\right]' &= 1 - \mathfrak{T}(r) \quad \Rightarrow \\ [r - r_M\beta(r)]' &= 1 - \mathfrak{T}(r) \quad \Rightarrow \\ 1 - r_M\beta'(r) &= 1 - \mathfrak{T}(r) \quad \Rightarrow \\ \mathfrak{T}(r) &= r_M\beta'(r), \end{aligned} \tag{54}$$

where we used our ansatz, and which therefore determines $\mathfrak{T}(r)$ in terms of $\beta(r)$. Note that, since $\mathfrak{T}(r)$ is proportional to r^2 times the energy density, and must therefore be positive, we must have that $\beta'(r) \geq 0$ for all values of r where the matter is located, and in fact everywhere. From the same t component of the field equation, shown in Equation (24), we can get directly the quantity $r\lambda'(r)$, since that equation can be written as

$$\begin{aligned} 1 - 2r\lambda'(r) &= e^{2\lambda(r)}\,[1 - \mathfrak{T}(r)] \\ &= \frac{r - r_M\,[r\beta'(r)]}{r - r_M\beta(r)} \quad \Rightarrow \\ 2r\lambda'(r) &= \frac{r - r_M\beta(r) - r + r_M r\beta'(r)}{r - r_M\beta(r)} \\ &= \frac{-r_M\beta(r) + r_M r\beta'(r)}{r - r_M\beta(r)} \quad \Rightarrow \\ [r\lambda'(r)] &= -\frac{r_M}{2}\,\frac{\beta(r) - [r\beta'(r)]}{r - r_M\beta(r)}, \end{aligned} \tag{55}$$

where we again used our ansatz, as well as the solution for $\mathfrak{T}(r)$ given in Equation (54), and which therefore determines $\lambda'(r)$ in terms of $\beta(r)$. Using now, once more, that result for $\mathfrak{T}(r)$ and the r component of the field equation, shown in Equation (25), we can get the quantity $r\nu'(r)$, since that equation can be written as

$$\begin{aligned}
1 + 2r\nu'(r) &= \mathrm{e}^{2\lambda(r)} \left[1 + \omega\mathfrak{T}(r)\right] \\
&= \frac{r + \omega r_M \left[r\beta'(r)\right]}{r - r_M\beta(r)} \quad \Rightarrow \\
2r\nu'(r) &= \frac{r + \omega r_M r\beta'(r) - r + r_M\beta(r)}{r - r_M\beta(r)} \\
&= \frac{\omega r_M r\beta'(r) + r_M\beta(r)}{r - r_M\beta(r)} \quad \Rightarrow \\
\left[r\nu'(r)\right] &= \frac{r_M}{2}\,\frac{\beta(r) + \omega\left[r\beta'(r)\right]}{r - r_M\beta(r)},
\end{aligned} \tag{56}$$

where we once again used our ansatz, and which therefore determines $\nu'(r)$ in terms of $\beta(r)$. Note that $\nu'(r)$ is *not* equal to $-\lambda'(r)$, as would be the case for the Schwarzschild solution. However, it is to be expected that $\nu'(r)$ has $-\lambda'(r)$ as its $r \to \infty$ asymptotic limit. With this we have two of the three quantities that appear in the left-hand side of the θ component of the field equation, shown in Equation (26). Since we have that

$$r\left[r\nu'(r)\right]' = r^2\nu''(r) + r\nu'(r), \tag{57}$$

and using once again the result for $\mathfrak{T}(r)$ given in Equation (54), the θ component equation given in Equation (26) can now be written in the form

$$r\left[r\nu'(r)\right]' + \left[r\nu'(r)\right]^2 - \left[r\lambda'(r)\right]\left[r\nu'(r)\right] = \left[r\lambda'(r)\right] + \mathrm{e}^{2\lambda(r)}\omega r_M\beta'(r). \tag{58}$$

In order to be able to write out this equation, we must now calculate in terms of $\beta(r)$ the quantity $r\left[r\nu'(r)\right]'$. We can do this by simply differentiating the quantity $\left[r\nu'(r)\right]$. The detailed calculation can be found in Subsection A.2.1 of Appendix A, and the result is

$$r\left[r\nu'(r)\right]' = r_M\,\frac{\left(\begin{array}{c}\omega\left[r - r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\} + \omega r_M\left[r\beta'(r)\right]^2 + \\ +(1-\omega)r\left[r\beta'(r)\right] - r\beta(r)\end{array}\right)}{2\left[r - r_M\beta(r)\right]^2}, \tag{59}$$

which therefore determines $\nu''(r)$ in terms of $\beta(r)$. We now have all the quantities that appear in the left-hand side of the θ component of the field equation, shown in Equation (26). In fact, just as in the case of the vacuum solution, the solution in the presence of fluid matter is characterized by the following set of quantities, this time written in terms of $\beta(r)$,

$$\mathfrak{T}(r) = r_M\beta'(r), \tag{60}$$

$$\mathrm{e}^{2\lambda(r)} = \frac{r}{r - r_M\beta(r)}, \tag{61}$$

$$r\nu'(r) = \frac{r_M}{2}\,\frac{\beta(r) + \omega\left[r\beta'(r)\right]}{r - r_M\beta(r)}, \tag{62}$$

$$r\lambda'(r) = -\frac{r_M}{2}\,\frac{\beta(r) - \left[r\beta'(r)\right]}{r - r_M\beta(r)}, \tag{63}$$

$$r\left[r\nu'(r)\right]'(r)=\frac{r_M}{2}\,\frac{\begin{pmatrix}\omega\left[r-r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\}+\\ +\omega r_M\left[r\beta'(r)\right]^2+\\ +(1-\omega)r\left[r\beta'(r)\right]-r\beta(r)\end{pmatrix}}{\left[r-r_M\beta(r)\right]^2}. \tag{64}$$

Note that the quantity $\exp[2\nu(r)]$ remains undetermined, which does not really present a problem, since it does not appear in the components of the field equation. Once $\beta(r)$ is determined in each particular case, $\nu(r)$ can be obtained from $\nu'(r)$ by straightforward integration. These expressions are to be used in what follows in order to define the asymptotic conditions in the $r \to \infty$ asymptotic limit. This is what we will do next. Later on we will return to the discussion of the single equation yet to be satisfied, the θ component equation in the form shown in Equation (58).

On the one hand, in Section 3 we calculated from the vacuum solution the values of the $r \to \infty$ asymptotic limits, for the various corresponding quantities involved in the solution in the presence of fluid mater, asymptotic values that were given in Equations (47)—(51). On the other hand, in this section we calculated the solution in the presence of fluid matter, and listed what is essentially the same set of relevant quantities in Equations (60)—(64). We are now ready to discuss the corresponding asymptotic conditions. It is to be expected, of course, that they will result in corresponding asymptotic conditions on $\beta(r)$ and its derivatives. We start by discussing the asymptotic condition on $\lambda(r)$. As we already discussed before, the expression in Equation (48) can only be the asymptotic limit of the expression in Equation (61) if we have that

$$\lim_{r\to\infty}\beta(r)=1. \tag{65}$$

Note that we do not have to worry directly about the asymptotic condition on $\nu(r)$, because $\nu(r)$ itself does not appear in the component equations, which contain only its derivatives. Therefore, next we discuss the asymptotic condition on $\lambda'(r)$. Given that $\beta(r) \to 1$, the expression in Equation (50) can only be the asymptotic limit of the expression in Equation (63) if we have that

$$\lim_{r\to\infty}\left[r\beta'(r)\right]=0. \tag{66}$$

Next we discuss the asymptotic condition on $\nu'(r)$. It is quite clear that the conditions above on $\beta(r)$ and $\beta'(r)$ are sufficient to ensure that the expression in Equation (49) will be the asymptotic limit of the expression in Equation (62). Let us recall that, once we have $\nu'(r)$ in terms of a known function $\beta(r)$, we can obtain $\nu(r)$ from $\nu'(r)$ by straightforward integration. When doing this integration, the integration constant must be chosen so that $\nu(r)$, just like $\lambda(r)$, goes to zero for $r \to \infty$, of course. In that limit we also expect that $\nu(r) = -\lambda(r)$. Finally, we discuss the asymptotic condition on $\nu''(r)$. Given that $\beta(r) \to 1$ and that $[r\beta'(r)] \to 0$, the expression in Equation (51) can only be the asymptotic limit of the expression in Equation (64) if we have that

$$\lim_{r\to\infty}\left[r^2\beta''(r)\right]=0. \tag{67}$$

Therefore, we have the complete set of asymptotic conditions to be satisfied by $\beta(r)$ and its derivatives,

$$\begin{aligned}\lim_{r\to\infty}\beta(r) &= 1,\\ \lim_{r\to\infty}\left[r\beta'(r)\right] &= 0,\\ \lim_{r\to\infty}\left[r^2\beta''(r)\right] &= 0.\end{aligned} \tag{68}$$

We must now discuss what happens near $r = 0$. At this point we will have conditions associated to the regularity of the energy density. If we consider that, according to the definitions in Section 2, the energy density is given by

$$T_0(r) = \frac{\mathfrak{T}(r)}{\kappa r^2}, \tag{69}$$

where we also have that $\mathfrak{T}(r) = r_M \beta'(r)$, as shown in Equation (54), it follows that we have

$$T_0(r) = \frac{r_M}{\kappa} \frac{\beta'(r)}{r^2}. \tag{70}$$

If the energy density $T_0(r)$ is to be non-singular at $r = 0$, and have a limited integral around that point, then we must have that the limit

$$\lim_{r \to 0} \frac{\beta'(r)}{r^2} \tag{71}$$

exists and is finite. This implies that at least for the first derivative of $\beta(r)$ we must have that $\beta'(0) = 0$. In fact, since we must also have that $\beta'(r)/r \to 0$ when $r \to 0$, one can show that the same has to be true for the second derivative as well, that is, we also have that $\beta''(0) = 0$. This leads us to a picture of a function $\beta(r)$ that has both zero derivative and zero second derivative at both ends of the real r semi-axis.

One may also argue that it is necessary that $\beta(0)$ *not* be a strictly positive number. The argument leading to this condition is as follows. According to the motivation leading to the construction of our solution, the quantity $r_M \beta(r)$, where $r_M = 2MG/c^2$ and $\beta(r) \in [0, 1]$, is effectively a certain fraction of the total mass M. If $\beta(0) > 0$ then this quantity has a finite and non-zero positive limit when we make $r \to 0$. What this means is that there is a certain finite and non-zero mass, given by $r_M \beta(0)$, which is inside a sphere whose surface is at the radial position r, and that this holds for *all* r. However, this means that for some value of r this finite and non-zero mass will be inside the Schwarzschild radius associated to itself, thus leading to the existence of an event horizon within the matter distribution, which contradicts our hypotheses here. In order to avoid this, and *assuming* that $\beta(r) \in [0, 1]$, we might be tempted to conclude that we must have that

$$\lim_{r \to 0} \beta(r) = 0, \tag{72}$$

that is, that we must have $\beta(0) = 0$. However, we should not take this heuristic motivation too seriously, and while it seems inevitably true that we cannot have $\beta(0) > 0$, we also *cannot* definitely assert that we must have $\beta(0) = 0$, because there is really no reason why $\beta(r)$ cannot be *negative*. Instead, once the ansatz given in Equation (53) is assumed, we must then follow wherever the field equation takes us. In fact, contrary to what the heuristic intuition may seem to indicate, $\beta(0)$ is indeed *always* negative. When $\beta(r)$ becomes negative it becomes impossible for the factor $\exp[2\lambda(r)]$ to diverge, as it does at the event horizon of the Schwarzschild solution, and this makes the solution regular and avoids the existence of event horizons within the matter distribution. Considering the fact that, due to the positivity of the energy, we must have $\beta'(r) \geq 0$ for all r, and taking into account the asymptotic conditions derived here for $\beta(r)$ and its derivatives, as well as the conditions at $r = 0$, we can state that the function $\beta(r)$ must have a very simple qualitative behavior, going monotonically from some negative value at $r = 0$ to the value 1 for $r \to \infty$. We have therefore the complete set of relevant conditions at the two ends of the real r semi-axis,

$$\begin{aligned}
\lim_{r\to 0}\beta'(r) &= 0,\\
\lim_{r\to 0}\beta''(r) &= 0,\\
\lim_{r\to\infty}\beta(r) &= 1,\\
\lim_{r\to\infty}\left[r\beta'(r)\right] &= 0,\\
\lim_{r\to\infty}\left[r^2\beta''(r)\right] &= 0.
\end{aligned} \tag{73}$$

Note that, since we have here a second-order field equation, which will give origin to a second-order equation for $\beta(r)$, we in fact can satisfy only *two* independent conditions using the corresponding integration constants. Therefore any additional conditions must arise as automatic consequences of the first two. As we will see, it is possible to reduce the equation for $\beta(r)$ to a form in which one has only two free parameters to deal with, both physically meaningful, one of which will be given by the total mass M present, leading to the parameter r_M, and only one, the parameter ω, which directly affects the integration process itself. Since we have fewer parameters to adjust than conditions to meet, the problem can only be solved if there are internal consistency structures within the system that guarantee that most conditions are automatically met when we adjust the parameters in order to satisfy what conditions we can. We will see that this is indeed the case.

5 Final Component Equation

Up to this point in our development all the quantities related to our new solution have been left written in terms of the single function $\beta(r)$. The only equation still to be satisfied is the θ component of the field equation, shown in Equation (58), the one that, therefore, will become the equation that determines $\beta(r)$. The rather long detailed calculations needed in order to write that equation explicitly in terms of $\beta(r)$, as well as its subsequent simplification, can be found in Subsection A.2.2 of Appendix A. The result is a rather complicated non-linear differential equation that, together with the limiting conditions discussed before, determines the function $\beta(r)$, for all r, and that for convenience we will write with an extra overall factor of r_M, as

$$\begin{aligned}
2\omega r_M\left[r - r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\} + \omega(1+\omega)r_M^2\left[r\beta'(r)\right]^2 + &\\
-6\omega r r_M\left[r\beta'(r)\right] + (1+7\omega)r_M^2\beta(r)\left[r\beta'(r)\right] &= 0,
\end{aligned} \tag{74}$$

with the boundary conditions at $r=0$ and the asymptotic conditions for $r\to\infty$ given by

$$\begin{aligned}
\beta'(0) &= 0,\\
\beta''(0) &= 0,\\
\lim_{r\to\infty}\beta(r) &= 1,\\
\lim_{r\to\infty}\left[r\beta'(r)\right] &= 0,\\
\lim_{r\to\infty}\left[r^2\beta''(r)\right] &= 0.
\end{aligned} \tag{75}$$

It is an interesting fact that this same equation for $\beta(r)$, given in Equation (74), can also be derived from the consistency condition shown in Equation (27). This serves as an independent verification of the correctness of the derivation mentioned above and detailed in Appendix A. The detailed calculation for this second derivation can be found in Subsection A.2.3 of Appendix A. If we use in the consistency condition the results obtained before for $\nu'(r)$ and $\mathfrak{T}(r)$, we find that it becomes exactly the same expression shown in Equation (74). Therefore, once we have determined $\beta(r)$ we will have satisfied *all* the relevant equations, including all the component equations and the consistency condition as well.

We can simplify Equation (74) somewhat by means of a simple change of variables. For this purpose we define a new dimensionless radial variable by $\xi = r/r_0$ where r_0 is simply an arbitrary reference point for measuring the radial positions. We then have the particular value $\xi_M = r_M/r_0$, which corresponds therefore to the total mass M. It would also be possible to define another particular value, given by $\xi_S = r_S/r_0$, but this value will not appear in the equations, due to the difficulty in defining in general, and with precision, the position r_S of the sphere that contains essentially all the fluid matter. It is not difficult to see that the definition of ξ implies for the corresponding derivatives that we have

$$r\partial_r = \xi\partial_\xi. \tag{76}$$

With this, and using, from now on, the primes to denote derivatives with respect to ξ, we can write our final equation in the form

$$\begin{aligned} 2\omega\xi_M\left[\xi - \xi_M\beta(\xi)\right]\left\{\xi\left[\xi\beta'(\xi)\right]'\right\} + \omega(1+\omega)\xi_M^2\left[\xi\beta'(\xi)\right]^2 + & \\ -6\omega\xi\xi_M\left[\xi\beta'(\xi)\right] + (1+7\omega)\xi_M^2\beta(\xi)\left[\xi\beta'(\xi)\right] &= 0. \end{aligned} \tag{77}$$

We may further simplify this equation by defining a new dimensionless function $\gamma(\xi)$, in terms of the equally dimensionless function $\beta(\xi)$, by

$$\begin{aligned} \gamma(\xi) &= \frac{r_M}{r_0}\beta(\xi) \\ &= \xi_M\beta(\xi). \end{aligned} \tag{78}$$

Therefore, we get for our final equation, now for the function $\gamma(\xi)$,

$$\begin{aligned} 2\omega\left[\xi - \gamma(\xi)\right]\left\{\xi\left[\xi\gamma'(\xi)\right]'\right\} + \omega(1+\omega)\left[\xi\gamma'(\xi)\right]^2 + & \\ -6\omega\xi\left[\xi\gamma'(\xi)\right] + (1+7\omega)\gamma(\xi)\left[\xi\gamma'(\xi)\right] &= 0, \end{aligned} \tag{79}$$

where we see that the parameter ξ_M no longer appears in the equation, which now depends only on the parameter ω. On the other hand, ξ_M now appears in one of the relevant boundary and asymptotic conditions on $\gamma(\xi)$. The boundary conditions at $\xi = 0$ and the $\xi \to \infty$ asymptotic conditions for $\gamma(\xi)$ are given by

$$\begin{aligned} \gamma'(0) &= 0, \\ \gamma''(0) &= 0, \\ \lim_{\xi\to\infty}\gamma(\xi) &= \xi_M, \\ \lim_{\xi\to\infty}\left[\xi\gamma'(\xi)\right] &= 0, \\ \lim_{\xi\to\infty}\left[\xi^2\gamma''(\xi)\right] &= 0. \end{aligned} \tag{80}$$

Note that with these changes of variable we have clearly separated the roles of the parameter ξ_M, which represents here the total mass M, and of the parameter ω, which represents here the state of the fluid matter. While the parameter ω appears in the differential equation and therefore modulates the propagation of the solution along ξ, the parameter ξ_M appears only in the asymptotic condition involving $\gamma(\xi)$. We may now write all our previous results in terms of $\gamma(\xi)$, rather that $\beta(r)$,

$$\mathfrak{T}(\xi) = \gamma'(\xi), \tag{81}$$

$$e^{2\lambda(\xi)} = \frac{\xi}{\xi - \gamma(\xi)}, \tag{82}$$

$$[\xi\nu'(\xi)] = \frac{1}{2}\frac{\gamma(\xi) + \omega\,[\xi\gamma'(\xi)]}{\xi - \gamma(\xi)}, \tag{83}$$

$$[\xi\lambda'(\xi)] = -\frac{1}{2}\frac{\gamma(\xi) - [\xi\gamma'(\xi)]}{\xi - \gamma(\xi)}, \tag{84}$$

$$\xi\,[\xi\nu'(\xi)]'(\xi) = \frac{1}{2}\,\frac{\left(\begin{array}{c}\omega\,[\xi-\gamma(\xi)]\,\{\xi\,[\xi\gamma'(\xi)]'\} + \\ +\omega\,[\xi\gamma'(\xi)]^2 + \\ +(1-\omega)\xi\,[\xi\gamma'(\xi)] - \xi\gamma(\xi)\end{array}\right)}{[\xi-\gamma(\xi)]^2}. \tag{85}$$

For use in the arguments that follow, it is important to record here that Equation (79) can also be written in the form

$$\begin{aligned} 2\omega\xi^2\,[\xi-\gamma(\xi)]\,\gamma''(\xi) + \omega(1+\omega)\xi^2\,[\gamma'(\xi)]^2 + & \\ -4\omega\xi^2\gamma'(\xi) + (1+5\omega)\xi\gamma(\xi)\gamma'(\xi) &= 0, \end{aligned} \tag{86}$$

where we have used the fact that

$$\xi\,[\xi\gamma'(\xi)]' = \xi^2\gamma''(\xi) + \xi\gamma'(\xi). \tag{87}$$

We must now consider how to solve the equation for the function $\beta(r)$ that appears in the invariant interval, or equivalently the equation for $\gamma(\xi)$ given in Equations (79) or (86). For the time being the complete determination of $\gamma(\xi)$ has to be made numerically, but enough can be established about the properties of the solutions to give us a fairly complete picture of the physics involved. Given the general qualitative behavior of the function $\beta(r)$, which we already know, and given the simple relation between $\beta(r)$ and $\gamma(\xi)$, we can state at once that the function $\gamma(\xi)$ must have the general qualitative behavior shown in Figures 2 and 3. Just like $\beta(r)$, the function $\gamma(\xi)$ must have non-negative derivative everywhere, must have a finite negative value at $\xi = 0$ and a finite positive limit when $\xi \to \infty$.

Certain special values of ξ play an important role in this analysis. As shown in the figures, ξ_i is the point of inflection of the function $\gamma(\xi)$, where we have that $\gamma''(\xi_i) = 0$, and which will play an important role in the description and classification of the solutions. The point ξ_r is the root of the function $\gamma(\xi)$, the single point where we have that $\gamma(\xi_r) = 0$. The critical point ξ_c is the point where $\exp[2\lambda(\xi)]$ may develop a singularity, and it is the point which may become an event horizon in a certain limit. It is defined as the point where $\gamma'(\xi_c) = 1$, and it is also the point where the difference $[\xi - \gamma(\xi)]$ has its minimum value. This is a consequence of the fact that the singularity in the metric factor $\exp[2\lambda(\xi)]$ comes about when the graph of $\gamma(\xi)$ just touches the graph of the identity function, so that $[\xi_c - \gamma(\xi_c)]$ becomes zero.

We know that the function $\gamma(\xi)$ has non-negative derivative at all points, and that it goes from some negative finite value at $\xi = 0$ to some positive finite value for $\xi \to \infty$. It follows that the function cannot have any local maxima or minima and that it must have at least one inflection point. In fact, as we will see in what follows, it has exactly one inflection point. At this inflection point we have that $\gamma''(\xi_i) = 0$, and that the functions $\gamma(\xi_i)$ and $\gamma'(\xi_i)$ have some definite values. In principle, since the differential equation determining $\gamma(\xi)$ is of the second order, given the values of these two functions at this point, a solution for $\gamma(\xi)$ is completely determined. Interestingly, we can easily obtain from the equation itself

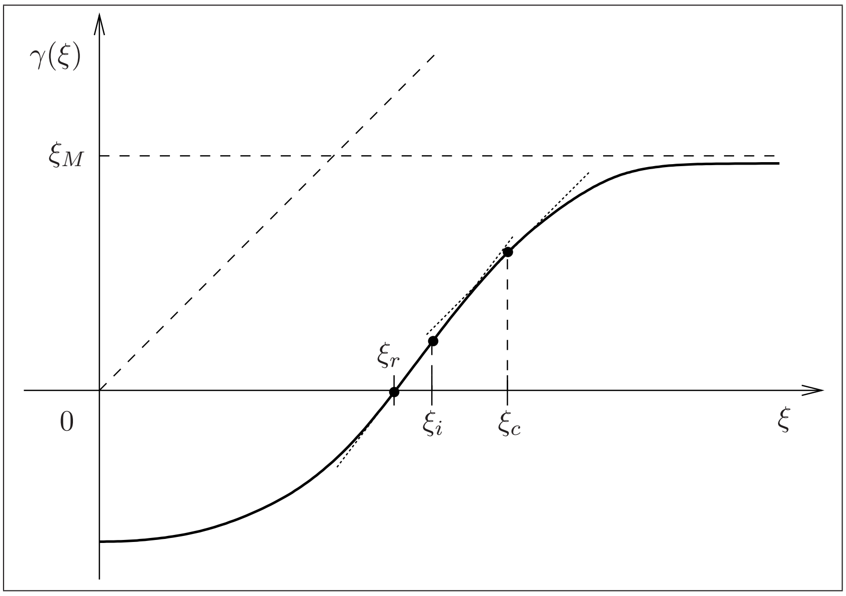

Figure 2: The qualitative behavior of the dimensionless function $\gamma(\xi)$, showing the main features: the inflection point ξ_i, the root ξ_r, the critical point ξ_c and the asymptotic limit ξ_M; the identity function ξ is also shown; the configuration show is that of the middle-density regime.

a definite relation between these two values, leaving us with only one arbitrary parameter for the determination of the solution. We can write an equation for the inflection point ξ_i of the function $\gamma(\xi)$, by simply putting $\gamma''(\xi_i) = 0$ in Equation (86), which then results in

$$\Big[- 4\omega\xi_i^2 + (1 + 5\omega)\xi_i\gamma(\xi_i) + \omega(1 + \omega)\xi_i^2\gamma'(\xi_i)\Big]\,\gamma'(\xi_i) = 0. \tag{88}$$

Since at the inflection point $\xi_i \neq 0$, this implies that either we have that $\gamma'(\xi_i) = 0$, which in fact only happens at $\xi = 0$ and for $\xi \to \infty$, or we must have that

$$\omega(1 + \omega)\xi_i\gamma'(\xi_i) = 4\omega\xi_i - (1 + 5\omega)\gamma(\xi_i). \tag{89}$$

This equation involves ω, ξ_i, $\gamma(\xi_i)$ and $\gamma'(\xi_i)$. Given values of ω and ξ_i, it relates the values of $\gamma(\xi_i)$ and $\gamma'(\xi_i)$ for a solution which has its inflection point at ξ_i. We may therefore describe and classify all the possible solution of this equation by the use of two parameters, one being the parameter ω that describes the state of the matter. We will choose the other to be the positive parameter given by $\pi(\xi_i) = \gamma'(\xi_i)$, and then the value $\gamma(\xi_i)$ is given by

$$\gamma(\xi_i) = \omega\xi_i\,\frac{4 - (1 + \omega)\pi(\xi_i)}{1 + 5\omega}. \tag{90}$$

Note that there is a single solution for this quantity, and therefore a single inflection point. Note also that the actual value of ξ_i can be chosen arbitrarily because, since we have that $\xi = r/r_0$, choosing this value just corresponds to choosing a position for the arbitrary reference point r_0. We now draw attention to the fact that the linear function

$$\gamma_0(\xi) = \frac{4\omega}{1 + 6\omega + \omega^2}\,\xi \tag{91}$$

is a particular solution of the equation that determines $\gamma(\xi)$, show in Equation (86). However, this solution clearly does *not* satisfy the correct asymptotic conditions. Is we consider

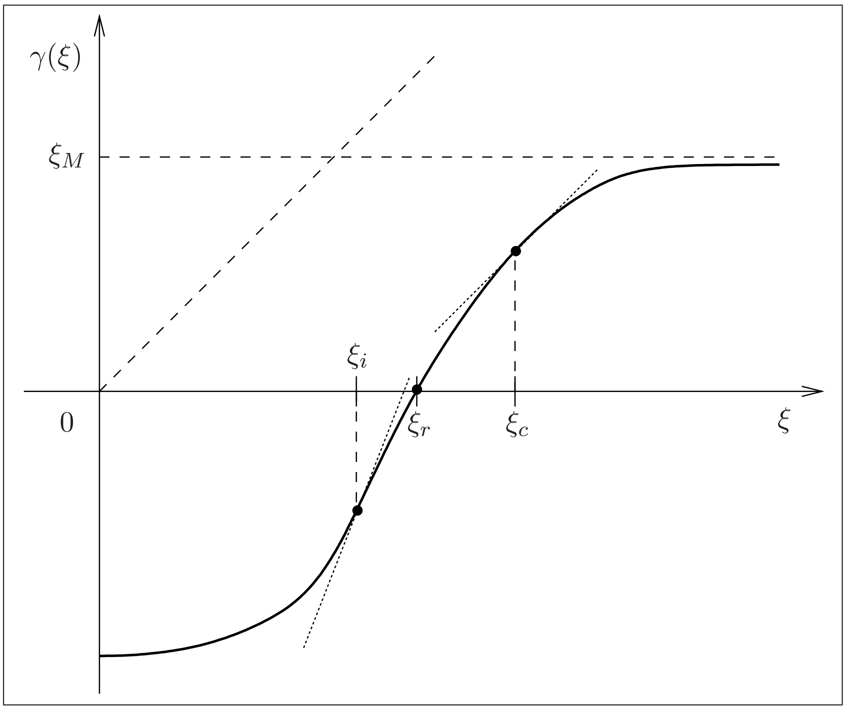

Figure 3: The qualitative behavior of the dimensionless function $\gamma(\xi)$, showing the main features: the root ξ_r, the inflection point ξ_i, the critical point ξ_c and the asymptotic limit ξ_M; the identity function ξ is also shown; the configuration show is that of the high-density regime.

the inflection point of $\gamma(\xi)$, which corresponds to the maximum value of the derivative $\gamma'(\xi)$, and where $\gamma''(\xi)$ is zero, then we must also impose that the second derivative of $\gamma'(\xi)$ be negative there, thus characterizing a local maximum of the derivative $\gamma'(\xi)$. This is therefore a condition on the third derivative of $\gamma(\xi)$, and we may calculate it explicitly using the form of that equation shown in Equation (86), which can be written as

$$\begin{aligned} 2\omega\left[\xi^2-\xi\gamma(\xi)\right]\gamma''(\xi)+\omega(1+\omega)\xi\left[\gamma'(\xi)\right]^2 &+ \\ -4\omega\xi\gamma'(\xi)+(1+5\omega)\gamma(\xi)\gamma'(\xi) &= 0. \end{aligned} \tag{92}$$

If we simply differentiate this equation, we get

$$\begin{aligned} 2\omega\left[\xi^2-\xi\gamma(\xi)\right]\gamma'''(\xi)+2\omega\left[2\xi-\gamma(\xi)-\xi\gamma'(\xi)\right]\gamma''(\xi)&+ \\ +\omega(1+\omega)\left[\gamma'(\xi)\right]^2+\omega(1+\omega)\xi\,2\gamma'(\xi)\gamma''(\xi)&+ \\ -4\omega\gamma'(\xi)-4\omega\xi\gamma''(\xi)&+ \\ +(1+5\omega)\left[\gamma'(\xi)\right]^2+(1+5\omega)\gamma(\xi)\gamma''(\xi) &= 0. \end{aligned} \tag{93}$$

Applying this at the inflection point, where we have that $\gamma''(\xi_i)=0$, we are left with

$$\begin{aligned} 2\omega\left[\xi_i^2-\xi_i\gamma(\xi_i)\right]\gamma'''(\xi_i)+\omega(1+\omega)\left[\pi(\xi_i)\right]^2 &+ \\ -4\omega\pi(\xi_i)+(1+5\omega)\left[\pi(\xi_i)\right]^2 &= 0 \quad\Rightarrow \\ 2\omega\left[\xi_i^2-\xi_i\gamma(\xi_i)\right]\gamma'''(\xi_i)-4\omega\pi(\xi_i)&+ \\ +\left(1+6\omega+\omega^2\right)\left[\pi(\xi_i)\right]^2 &= 0. \end{aligned} \tag{94}$$

Isolating the third derivative we get

$$\gamma'''(\xi_i) = \frac{4\omega - \left(1 + 6\omega + \omega^2\right)\pi(\xi_i)}{2\omega\xi_i\left[\xi_i - \gamma(\xi_i)\right]}\,\pi(\xi_i). \tag{95}$$

All the quantities appearing on this expression except the numerator are manifestly strictly positive at the inflection point, so that in order for the third derivative to be strictly negative we must impose that the numerator be strictly negative, thus leading to

$$\pi(\xi_i) > \frac{4\omega}{1 + 6\omega + \omega^2}. \tag{96}$$

The particular solution $\gamma_0(\xi)$ seems to represent a situation in which the gravitational field and the pressure are such as to cause the matter to escape the gravitational attraction well and thus spread out to infinity. It is not difficult to determine that for this particular solution we have that $\mathfrak{T}(\xi)$ is in fact a constant, corresponding to an energy density $T_0(r)$ that goes to zero at infinity slowly, as $1/r^2$, rather than exponentially fast. Therefore, in this case there is no sphere at some radial coordinate r_S that contains essentially all the matter. In other words, the matter fails to be localized. There seems to be no independent solutions for $\gamma(\xi)$ if $\pi(\xi_i)$ is smaller that this limiting value. This means, of course, that in this case there can be no static solution of the field equation. We have therefore the first and most important limit of our energy-density parameter $\pi(\xi_i)$,

$$\pi(\xi_i) > \frac{4\omega}{1 + 6\omega + \omega^2}. \tag{97}$$

We now shift our attention to the relation in Equation (90), that determines the value of $\gamma(\xi)$ at the inflection point. We observe that the sign of $\gamma(\xi_i)$ will be determined by the value of $\pi(\xi_i)$. Since all the other factors in the right-hand side of that equation are positive, we conclude that we will have $\gamma(\xi_i) \geq 0$ when

$$\pi(\xi_i) \leq \frac{4}{1 + \omega}. \tag{98}$$

The value given by the equality corresponds, of course, to $\gamma(\xi_i) = 0$, which means that the inflection point ξ_i coincides with the root ξ_r of $\gamma(\xi)$. In a complementary way, we will have $\gamma(\xi_i) \leq 0$ when

$$\pi(\xi_i) \geq \frac{4}{1 + \omega}. \tag{99}$$

There is therefore an interval of values of $\pi(\xi_i)$ given by

$$\frac{4\omega}{1 + 6\omega + \omega^2} < \pi(\xi_i) \leq \frac{4}{1 + \omega}, \tag{100}$$

for which $\gamma(\xi_i) \geq 0$. For the allowed values of ω the left limit is in the interval $(0, 3/7]$, and the right limit is in the interval $[3, 4)$, which are two intervals that do not intersect. In this case, since $\gamma(\xi_i) \geq 0$, the root ξ_r of $\gamma(\xi)$ is necessarily to the left of the inflection point ξ_i. This is the situation depicted in Figure 2. On the other hand, the critical point ξ_c will only exist if $\pi(\xi_i) \geq 1$, because otherwise, since $\pi(\xi_i)$ is the largest value of the derivative, it will never be equal to one, so that the critical point, which is defined as the point ξ_c where $\pi(\xi_c) = 1$, will not exist. The complementary interval of values of $\pi(\xi_i)$, to the one given in the equation above, is that for which we have that $\gamma(\xi_i) \leq 0$, and which is given by

$$\frac{4}{1 + \omega} \leq \pi(\xi_i) < \infty. \tag{101}$$

In this case, since we always have $\pi(\xi_i) \geq 1$, the critical point always exists, and the root ξ_r of $\gamma(\xi)$ is necessarily to the right of the inflection point ξ_i. This is the situation depicted in Figure 3. We may therefore classify the possible values of $\pi(\xi_i)$, and the corresponding possible solutions for $\gamma(\xi)$, in the following way.

Low Energy-Density Regime: if we have that

$$\frac{4\omega}{1+6\omega+\omega^2} < \pi(\xi_i) < 1, \tag{102}$$

then there is *no* critical point ξ_c, and we have that $\xi_r < \xi_i$. We will call this the *low energy-density regime*.

Middle Energy-Density Regime: if we have that

$$1 \leq \pi(\xi_i) \leq \frac{4}{1+\omega}, \tag{103}$$

then there is a critical point ξ_c, and we have that $\xi_r \leq \xi_i < \xi_c$. We will call this the *middle energy-density regime*.

High Energy-Density Regime: if we have that

$$\frac{4}{1+\omega} \leq \pi(\xi_i), \tag{104}$$

then there is a critical point ξ_c, and we have that $\xi_i < \xi_r < \xi_c$. We will call this the *high energy-density regime*.

Black-Hole Limit: the limit in which we make $\pi(\xi_i) \to \infty$ we will name the *black hole limit*, since it can be shown that in this limit, as seen from outside the horizon, the general character of the solutions does in fact approach that of a black hole.

Note that the case of the middle energy-density regime includes the special case

$$\pi(\xi_i) = \frac{4}{1+\omega}, \tag{105}$$

in which case we have $\gamma(\xi_i) = 0$, so that the points ξ_r and ξ_i coincide. Note also that in the case of the high energy-density regime we can easily prove that $\gamma(0)$ must be negative. Since we have that at the inflection point $\gamma(\xi_i) < 0$, and since we also have that the derivative $\gamma'(\xi)$ is always positive, it follows that at every point to the left of ξ_i the function $\gamma(\xi)$ must be smaller than $\gamma(\xi_i)$, and hence negative, including at the point $\xi = 0$.

Finally, note that the classification of the solutions in these various regimes does not mean that the solutions always exist, for every pair of values of the physical parameters ω and $\pi(\xi_i)$ within each one of the regimes. They may fail to exist, mostly for the larger values of ω, and in particular for the case $\omega = 1/3$, that corresponds to pure radiation, since in this case we have a maximum intensity of the expanding tendency of the pressure, as compared to the compressing tendency of gravity, thus leading to the possibility of the fluid matter being non-localized. The way in which a solution fails to exist is that the asymptotic conditions fail to hold. In particular, the $\xi \to \infty$ limit of $\gamma(\xi)$ fails to be finite, and instead increases without bound, much like what happens in the particular solution $\gamma_0(\xi)$ shown in Equation (91). In other words, unlike what is the typical situation for the case of linear differential equations, in this case the available parameters of the model cannot always be used to adjust the boundary conditions. In fact, most often they cannot be used in this way. The situation is simply that there are pairs of values of the two physical parameters ω and $\pi(\xi_i)$ for which a solution exists, and other for which a solution does not exist.

6 A Sampling of Numerical Results

Let us now show and briefly discuss some sample results obtained by means of a numerical computer program written to calculate the solution for $\gamma(\xi)$, so as to confirm the existence of solutions, as well as to confirm and display some of the main properties of the solutions, in order to illustrate their general behavior in a qualitative and visual way. A full description and discussion of the numerical solution, as well as of the properties of the solutions, will be given in a separate paper, which is forthcoming. This will include a detailed analysis of the $\pi(\xi_i) \to \infty$ black-hole limit.

In the graphs shown in Figures 4 to 6 one can see results for $\gamma(\xi)$ (solid line), $\pi(\xi)$ (dashed line) and $\pi'(\xi)$ (dotted line) for several runs of a program written to solve Equation (86). The program used to plot these graphs employed the Runge-Kutta fourth-order integration algorithm, running in quadruple-precision mode, that is, in double-precision mode on a 64-bit machine, with a self-adjustable increment for ξ, and about 1000 plotting points. The most relevant physical data in each case is reported within each graph, such as the value of $\gamma(0)$ and the value of ξ_M. The graph in Figure 4 exemplifies the low energy-density regime, the one in Figure 5 the middle energy-density regime, and that in Figure 6 the high energy-density regime.

In two of these graphs, those in Figures 5 and 6, a rendering of the identity function is included (dash-dotted line), for comparison with $\gamma(\xi)$. Since the difference $[\xi - \gamma(\xi)]$ appears in the denominator of the factor $\exp[2\lambda(r)]$ that multiplies the term dr^2 in the invariant interval, the proximity between the two corresponding curves tells us how close we are to having a singularity in that factor. The critical point ξ_c is the point of greatest proximity between ξ and $\gamma(\xi)$ and thus indicates approximately where the corresponding event horizon would form, if and when that was the case. The critical point is also the point of minimum of the difference $[\xi - \gamma(\xi)]$, which seems to be always positive, as one would expect.

Due to the rather large values either set or obtained for some of these quantities, in the graphs shown in Figures 5 and 6 the curves for $\pi(\xi)$ and $\pi'(\xi)$ are normalized so that the maximum value of their amplitudes become 1. For the same reason, in the graph in Figure 6 the negative part of the data for $\gamma(\xi)$ is given in a logarithmic scale, using base-10 logs. What is actually plotted is the quantity $-\log_{10}[1 - \gamma(\xi)]$. This means that the negative part of the curve for $\gamma(\xi)$ in this graph has much larger absolute values than is immediately apparent. All the other parts of the curves shown are on a linear scale, including the positive part of the curve for $\gamma(\xi)$.

The basic qualitative behavior of both the function $\gamma(\xi)$ and the function $\pi(\xi)$ is thus confirmed by the numerical analysis. The limits both for $\xi \to 0$ and for $\xi \to \infty$ behave just as was predicted by our previous analysis. In particular, the Schwarzschild solution is indeed the $r \to \infty$ asymptotic limit of our solutions here, thus indicating that the matter is indeed localized, and in most cases quite strongly so, with an apparent exponential decay of the energy density for large values of ξ. The fact that $\gamma(\xi)$ and hence $\beta(r)$ can become negative and large has important consequences. The fact that they become negative seems to be a general feature of all the solutions worked out so far. Once $\gamma(\xi)$ and hence $\beta(r)$ become negative, the metric factor $\exp[2\lambda(r)]$ becomes

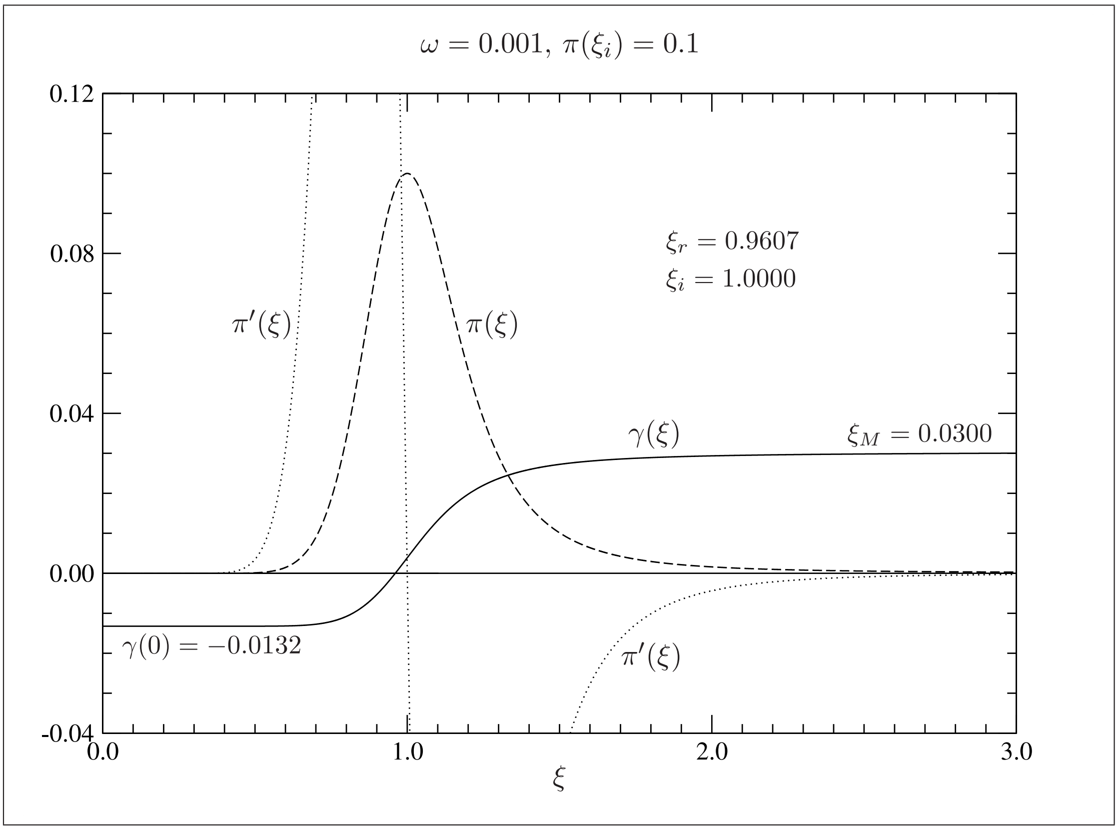

Figure 4: Sample solution for the function $\gamma(\xi)$ (solid line), showing also the the corresponding derivative function $\pi(\xi)$ (dashed line), as well as the second derivative function $\pi'(\xi)$ (dotted line). In this case we used $\omega = 0.001$ and $\pi(\xi_i) = 0.1$ at the inflection point, so that we are in the low energy-density regime.

$$e^{2\lambda(r)} = \frac{1}{1 + |\gamma(\xi)|/\xi}, \tag{106}$$

which goes towards zero as $|\gamma(\xi)|/\xi$ increases. Note that, once $\gamma(\xi)$ becomes negative, there is no longer any possibility of this factor having a singularity like the one at r_M in the Schwarzschild solution. When $|\gamma(\xi)|/\xi \gg 1$, we have for the physical element of length $d\ell$ in the radial direction

$$\begin{aligned} d\ell &= e^{\lambda(r)} dr \\ &= \frac{dr}{\sqrt{1 + |\gamma(\xi)|/\xi}} \\ &\ll dr. \end{aligned} \tag{107}$$

When $\xi \to 0$ we have that $\gamma(\xi)$ tends to some possibly large but finite negative value, so that we actually have that

$$\lim_{\xi \to 0} |\gamma(\xi)|/\xi = \infty, \tag{108}$$

which means that

$$\lim_{r \to 0} e^{2\lambda(r)} = 0. \tag{109}$$

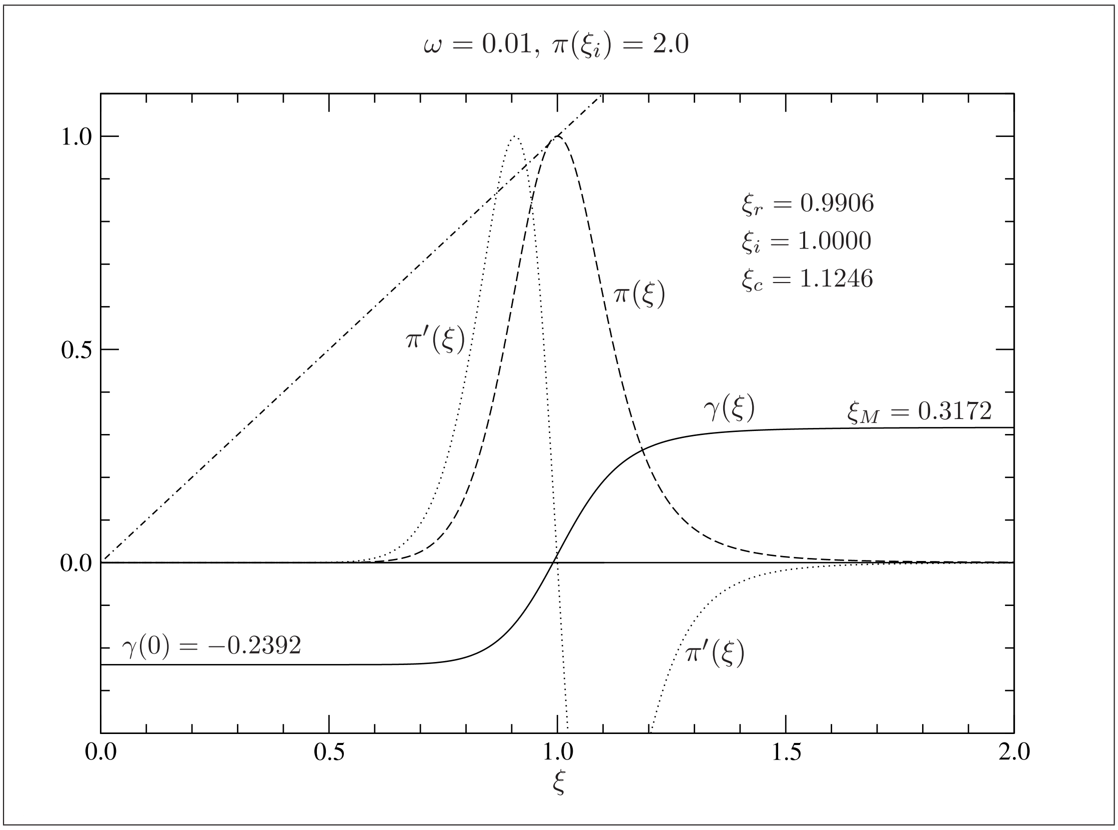

Figure 5: Sample solution for the function $\gamma(\xi)$ (solid line), showing also the identity function (dash-dotted line) and normalized versions of the corresponding derivative function $\pi(\xi)$ (dashed line) and second derivative function $\pi'(\xi)$ (dotted line), so that their maximum amplitudes become 1. In this case we used $\omega = 0.01$ and $\pi(\xi_i) = 2.0$ at the inflection point, so that we are in the middle energy-density regime.

The numerical data seems to indicate that, in the situations in which $[\xi_c - \gamma(\xi_c)]$ is close to zero at the critical point, the quantity $|\gamma(\xi)|$ seems to tend to become very large for smaller non-zero values of ξ, so that the radial lengths are significantly shrunk in most of the interior of the fluid matter distribution. The data seems to be consistent with the situation in which, in the limit in which we make $\pi(\xi_i) \to \infty$, we have that $[\xi_c - \gamma(\xi_c)] \to 0$, and also that $\gamma(0) \to -\infty$ linearly with $\pi(\xi_i)$. In fact, it seems that we also have that

$$\lim_{\pi(\xi_i)\to\infty} |\gamma(\xi)| = \infty, \tag{110}$$

for all values of ξ smaller than ξ_r, so that in this limit the radial lengths shrink all the way to zero in most of the interior of the region containing the matter. In this way, the approach to a black-hole configuration seems to be tied up with a complete shrinkage of most of the volume of the region where the fluid matter is located. Note that, due to the shrinking of the radial lengths, below the value ξ_r of ξ there is no possibility of working out an illustrative isometric embedding such as the one we worked out for the Schwarzschild solution, shown in Figure 1.

In such circumstances the geometry within the region containing the fluid matter distribution does have some rather odd characteristics indeed, since the radial lengths shrink while the angular lengths at the same position do not change at all. In essence, this is what

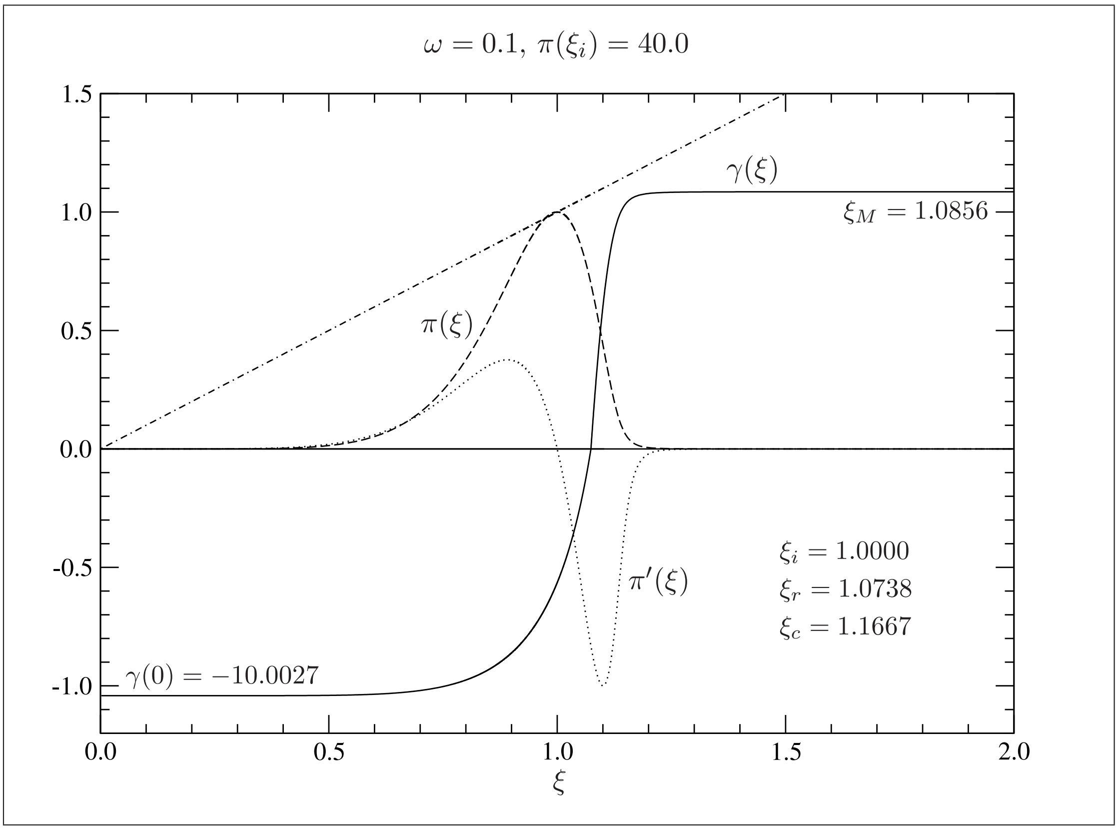

Figure 6: Sample solution for the function $\gamma(\xi)$ (solid line), showing also the identity function (dash-dotted line) and normalized versions of the corresponding derivative function $\pi(\xi)$ (dashed line) and second derivative function $\pi'(\xi)$ (dotted line), so that their maximum amplitudes become 1. In this case we used $\omega = 0.1$ and $\pi(\xi_i) = 40.0$ at the inflection point, so that we are in the high energy-density regime. In this case the negative portion of $\gamma(\xi)$ is shown in a base-10 logarithmic scale.

makes the embedding just mentioned impossible. Since it suffices for one of the lengths involved do decrease in order for the volume to decrease, this radial shrinkage does cause the volume to shrink under the fluid matter distribution. Note that, although the angular lengths do not change, they cease to be anything close to geodesics of the spatial geometry. One can always go from any point to any other point within the region containing the fluid matter distribution through only radial displacements, going from the first point to the center and from the center to the second point. With enough radial shrinkage, the distance traversed in this way will be smaller than that of a path between the two points that involves angular displacements. If the radial lengths shrink all the way to zero, then the geodesic distances between any two points in the shrunk region are zero. In effect, it is as if all the points in that region become effectively the same point.

According to the numerical data, the larger the value of $\pi(\xi_i)$, where ξ_i is the position of the inflection point of $\gamma(\xi)$, the closer to a singularity of $\exp[2\lambda(r)]$ at the critical point, where $[\xi_c - \gamma(\xi_c)] \to 0$, we get. But so far indications are that one never actually gets to such a singular solution for finite values of $\pi(\xi_i)$. The larger the value of $\pi(\xi_i)$, the closer r_M will be to the radial position r_S of the sphere that contains essentially all the fluid matter, coming from within. In the cases which are closer to exhibiting a singularity of $\exp[2\lambda(r)]$, and therefore closer to the formation of a black hole with an event horizon, the remaining internal volume seems to be highly concentrated near the surface of the matter distribution, while the radial lengths seem to shrink to zero somewhat faster for smaller values of ξ. In fact, under these conditions the energy density seems to develop a large and narrow peak near that surface, as can be seen in some cases, if one recalls that $\pi(\xi) = \mathfrak{T}(\xi)$ is closely related to the energy density. All these issues will be described and examined in detail in the aforementioned forthcoming paper.

The case $\omega = 1/3$ corresponds to "fluid matter" in the form of pure radiation, and it is interesting that even in this case, so long as $\pi(\xi_i)$ is large enough, we do still obtain static solutions, in which the fluid matter is still strongly localized. Of course, at least in part this is being allowed by the fact that we are ignoring losses of energy by outward radiation from the surface of the matter distribution, a more detailed treatment of which would certainly lead to non-static solutions. However, if we are close to a configuration that has a singularity of $\exp[2\lambda(r)]$ at the critical point, where $[\xi_c - \gamma(\xi_c)] \to 0$, then the strong red shift effect for any outward radiation from the vicinity of the spherical surface at that radial position will tend to make such energy losses very small, and then our static solution may still be a fair approximation of reality.

7 Conclusions and Outlook

We have established a two-parameter class of solutions of the Einstein gravitational field equation, corresponding to the presence of static fluid matter with the equation of state $P = \omega\rho$, for all values of ω in the relevant interval $(0, 1/3]$. There are in fact three parameters in play, the equation of state parameter ω, the Schwarzschild radius r_M and the radius r_S of the sphere containing essentially all the matter, where we could put our arbitrary radial reference point r_0, but the solutions depend only on ω and on a ratio such as r_M/r_S. The fluid matter is strongly localized, and the Schwarzschild solution is indeed the $r \to \infty$ asymptotic limit of the solutions we found here.

Two main physical facts can be abstracted from the results. The first is that the central space in the interior of a dense conglomeration of matter is necessarily shrunk, with actual radial lengths that are smaller, and in some circumstances much smaller, than the corresponding variations of the coordinate r. It is even possible that these lengths go all the way to zero in the limit in which a black hole with an event horizon would form. In any case, this reduces the available volume of space under the matter, and therefore compresses the matter even further. This seems to indicate that the apparent volume of a very dense star or black hole is therefore much larger than the actual physical one.

The second is that it is possible to have a static solution with a localized conglomeration of pure energy, that is, of matter with the pure radiation equation of state $\rho = 3P$, matter that consists, therefore, of pure relativistic radiation. Of course this is, in part, a consequence of the fact that we are ignoring the outward radiation of energy from the surface of the fluid matter to asymptotic infinity. However, close to the limit in which a black hole would form the loss of energy by this mechanism is strongly contained by the red shift effect upon outgoing radiation originating from close to the position of the Schwarzschild radius, thus justifying the static hypothesis as an approximation. Therefore, we can only claim that this type of static or quasi-static solution with pure radiation exists in the case of distributions of fluid matter which are very close to a black-hole configuration.

In the limit in which we make $\pi(\xi_i) \to \infty$, where ξ_i is the position of the inflection point of $\gamma(\xi)$, we seem to have that $r_S \to r_M$, where $r_S > r_M$ is the radial position of the sphere that contains essentially all the fluid matter, and r_M is the Schwarzschild radius of the corresponding mass M. At the same time we seem to have that $\gamma(\xi) \to -\infty$ for almost all $\xi < \xi_i$, so that all radial lengths within the matter distribution are going toward zero. In this limit one does seem to obtain a black hole, in the usual sense given to that term, or at least something that is indistinguishable from it when viewed from the outside. This is not, however, a "naked" black hole as is usually thought to be the case, but a "dressed" one, with plenty of matter and a definite, if unexpected, geometry within it.

It is very interesting to observe that this situation establishes an unexpected connection with G. 't Hooft's ideas about quantum mechanics and black holes [5]. In describing his studies involving quantum entanglement around black holes, that author has used the figure of speech that the interior of the black hole "is not really there", motivated by quantum correlations between antipodal points of the surface of the black hole. Well, one way to have this situation realized is to have the internal volume of the black hole be zero, and the distances between these diametrically opposed points also be zero, just like all other radial lengths within the region containing the fluid matter. It is interesting that while that author's conclusions come from a quantum theory involving black holes, our conclusions here emerge from a theory which is, in so far as one can currently see, entirely classical.

We end with a few words about the possible routes for the continuation of this line of work. It is to be noted that this is a very simple model, in so far as the hypotheses about the matter are concerned, and one should not expect it to have immediate applicability in all realistic physical cases of interest. For realistic stars, which are known to have an internal structure consisting of layers, it would probably be necessary to extend the model to include equations of state that depend on r, that is, to exchange the constant ω for a function $\omega(r)$. Also with the intent of making the model more realistic, a possibly simple extension would be to the time-dependent case, still with spherical symmetry. This would concern only the solution strictly within the matter, since by the Jebsen-Birkhoff [2, 3] theorem the Schwarzschild solution outside would not change in a significant way. It might be sufficient, though, to simply accommodate slowly changing solutions due to outward radiation from the surface of the fluid matter distribution.

It would certainly be interesting to extend the results obtained here to the case of the Kerr metric, so that the extended results would be applicable to rotating stars and black holes. However, this is likely to turn out to be quite a difficult enterprise. Another possible extension, as an alternative to having a fully non-homogeneous equation of state with a function $\omega(r)$, would be to fluid matter with a discretely non-homogeneous equation of state, possibly fluid matter with layered distributions, with different equations of state in each layer, each with a different constant ω, tied up together by the imposition of appropriate boundary conditions at the interfaces between layers. Although it is not at all clear that this is actually possible, it seems like a possibility worthwhile examining. Instabilities in these configurations could possibly lead to fast transitions such as those which are characteristic of some astrophysical phenomena.

Since what we presented here is a previously unknown class of solutions of the Einstein gravitational field equation, it is likely that these new solutions will find applications in the study of stars, as well as of other denser astrophysical objects. Further study of the non-linear differential equation show in Equations (77) for the function $\beta(\xi)$, and in Equations (79) and (86) for the function $\gamma(\xi)$, may turn out to be useful to elucidate the actual physical structure of extremely dense objects such as neutron stars and black holes. In particular, further numerical studies aimed at clarifying the behavior of various aspects of the geometry in the limit in which $\pi(\xi_i) \to \infty$, where ξ_i is the position of the inflection point of $\gamma(\xi)$, a limit which we might loosely describe as the *black-hole limit*, would probably be quite interesting. This is currently being worked on, and will be included in the aforementioned forthcoming paper.

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A Detailed Calculations

Here we present detailed and explicit versions of some of the calculations referred to in the text, and whose results are used there.

A.1 Consistency Condition

In this section we calculate this consistency condition for the specific case of time independence and spherical symmetry. In other words, we now write explicitly, under these conditions, the expression that is given in Equation (14) of the text. The energy-momentum tensor in Equation (13) of the text, in its mixed form, can be written symbolically in the following way,

$$T_\mu^\nu(r) = \delta_\mu^0 \delta_0^\nu T_0(r) + \delta_\mu^1 \delta_1^\nu T_1(r) + \delta_\mu^2 \delta_2^\nu T_2(r) + \delta_\mu^3 \delta_3^\nu T_3(r). \tag{111}$$

This symbolic form encodes the fact that $T_\mu^\nu(r)$ is a diagonal matrix. We may calculate the covariant divergence using this symbolic form of the tensor. In order to do this we first write explicitly the contracted covariant derivative of the mixed tensor $T_\mu^\nu(r)$ that appears in its covariant divergence,

$$D_\nu T_\mu^\nu(r) = \partial_\nu T_\mu^\nu(r) - \Gamma^\alpha{}_{\nu\mu} T_\alpha^\nu(r) + \Gamma^\nu{}_{\nu\alpha} T_\mu^\alpha(r). \tag{112}$$

As we will see, the only non-trivial condition that comes from Equation (14) of the text is the one for the case $\mu = 1$. However, let us calculate first the general, non-contracted case, of this covariant derivative,

$$D_\lambda T^\nu_\mu(r) = \partial_\lambda T^\nu_\mu(r) - \Gamma^\alpha{}_{\lambda\mu} T^\nu_\alpha(r) + \Gamma^\nu{}_{\lambda\alpha} T^\alpha_\mu(r). \tag{113}$$

We can do this using the symbolic form for $T^\nu_\mu(r)$,

$$\begin{aligned}
D_\lambda T^\nu_\mu(r) &= \partial_\lambda \left[\delta^0_\mu \delta^\nu_0 T_0(r) + \delta^1_\mu \delta^\nu_1 T_1(r) + \delta^2_\mu \delta^\nu_2 T_2(r) + \delta^3_\mu \delta^\nu_3 T_3(r)\right] + \\
&\quad -\Gamma^\alpha{}_{\lambda\mu} \left[\delta^0_\alpha \delta^\nu_0 T_0(r) + \delta^1_\alpha \delta^\nu_1 T_1(r) + \delta^2_\alpha \delta^\nu_2 T_2(r) + \delta^3_\alpha \delta^\nu_3 T_3(r)\right] + \\
&\quad +\Gamma^\nu{}_{\lambda\alpha} \left[\delta^0_\mu \delta^\alpha_0 T_0(r) + \delta^1_\mu \delta^\alpha_1 T_1(r) + \delta^2_\mu \delta^\alpha_2 T_2(r) + \delta^3_\mu \delta^\alpha_3 T_3(r)\right] \\
&= \delta^0_\mu \delta^\nu_0 [\partial_\lambda T_0(r)] + \delta^1_\mu \delta^\nu_1 [\partial_\lambda T_1(r)] + \delta^2_\mu \delta^\nu_2 [\partial_\lambda T_2(r)] + \delta^3_\mu \delta^\nu_3 [\partial_\lambda T_3(r)] + \\
&\quad -\Gamma^\alpha{}_{\lambda\mu} \delta^0_\alpha \delta^\nu_0 T_0(r) - \Gamma^\alpha{}_{\lambda\mu} \delta^1_\alpha \delta^\nu_1 T_1(r) - \Gamma^\alpha{}_{\lambda\mu} \delta^2_\alpha \delta^\nu_2 T_2(r) - \Gamma^\alpha{}_{\lambda\mu} \delta^3_\alpha \delta^\nu_3 T_3(r) + \\
&\quad +\Gamma^\nu{}_{\lambda\alpha} \delta^0_\mu \delta^\alpha_0 T_0(r) + \Gamma^\nu{}_{\lambda\alpha} \delta^1_\mu \delta^\alpha_1 T_1(r) + \Gamma^\nu{}_{\lambda\alpha} \delta^2_\mu \delta^\alpha_2 T_2(r) + \Gamma^\nu{}_{\lambda\alpha} \delta^3_\mu \delta^\alpha_3 T_3(r) \\
&= \delta^0_\mu \delta^\nu_0 \delta^1_\lambda T'_0(r) + \delta^1_\mu \delta^\nu_1 \delta^1_\lambda T'_1(r) + \delta^2_\mu \delta^\nu_2 \delta^1_\lambda T'_2(r) + \delta^3_\mu \delta^\nu_3 \delta^1_\lambda T'_3(r) + \\
&\quad -\Gamma^0{}_{\lambda\mu} \delta^\nu_0 T_0(r) - \Gamma^1{}_{\lambda\mu} \delta^\nu_1 T_1(r) - \Gamma^2{}_{\lambda\mu} \delta^\nu_2 T_2(r) - \Gamma^3{}_{\lambda\mu} \delta^\nu_3 T_3(r) + \\
&\quad +\Gamma^\nu{}_{\lambda 0} \delta^0_\mu T_0(r) + \Gamma^\nu{}_{\lambda 1} \delta^1_\mu T_1(r) + \Gamma^\nu{}_{\lambda 2} \delta^2_\mu T_2(r) + \Gamma^\nu{}_{\lambda 3} \delta^3_\mu T_3(r),
\end{aligned} \tag{114}$$

where we have used the fact that, since $T_\mu(r)$ are functions of only r, we have that

$$\partial_\lambda T_\mu(r) = \delta^1_\lambda T'_\mu(r), \tag{115}$$

where the prime indicates derivatives with respect to r, just as in the text. If we now contract the indices λ and ν we get

$$\begin{aligned}
D_\nu T^\nu_\mu(r) &= \delta^0_\mu \delta^\nu_0 \delta^1_\nu T'_0(r) + \delta^1_\mu \delta^\nu_1 \delta^1_\nu T'_1(r) + \delta^2_\mu \delta^\nu_2 \delta^1_\nu T'_2(r) + \delta^3_\mu \delta^\nu_3 \delta^1_\nu T'_3(r) + \\
&\quad -\Gamma^0{}_{\nu\mu} \delta^\nu_0 T_0(r) - \Gamma^1{}_{\nu\mu} \delta^\nu_1 T_1(r) - \Gamma^2{}_{\nu\mu} \delta^\nu_2 T_2(r) - \Gamma^3{}_{\nu\mu} \delta^\nu_3 T_3(r) + \\
&\quad +\Gamma^\nu{}_{\nu 0} \delta^0_\mu T_0(r) + \Gamma^\nu{}_{\nu 1} \delta^1_\mu T_1(r) + \Gamma^\nu{}_{\nu 2} \delta^2_\mu T_2(r) + \Gamma^\nu{}_{\nu 3} \delta^3_\mu T_3(r) \\
&= \delta^0_\mu \delta^1_0 T'_0(r) + \delta^1_\mu \delta^1_1 T'_1(r) + \delta^2_\mu \delta^1_2 T'_2(r) + \delta^3_\mu \delta^1_3 T'_3(r) + \\
&\quad -\Gamma^0{}_{0\mu} T_0(r) - \Gamma^1{}_{1\mu} T_1(r) - \Gamma^2{}_{2\mu} T_2(r) - \Gamma^3{}_{3\mu} T_3(r) + \\
&\quad +\Gamma^\nu{}_{\nu 0} \delta^0_\mu T_0(r) + \Gamma^\nu{}_{\nu 1} \delta^1_\mu T_1(r) + \Gamma^\nu{}_{\nu 2} \delta^2_\mu T_2(r) + \Gamma^\nu{}_{\nu 3} \delta^3_\mu T_3(r) \\
&= \delta^1_\mu T'_1(r) - \Gamma^0{}_{0\mu} T_0(r) - \Gamma^1{}_{1\mu} T_1(r) - \Gamma^2{}_{2\mu} T_2(r) - \Gamma^3{}_{3\mu} T_3(r) + \\
&\quad +\Gamma^\nu{}_{\nu 0} \delta^0_\mu T_0(r) + \Gamma^\nu{}_{\nu 1} \delta^1_\mu T_1(r) + \Gamma^\nu{}_{\nu 2} \delta^2_\mu T_2(r) + \Gamma^\nu{}_{\nu 3} \delta^3_\mu T_3(r),
\end{aligned} \tag{116}$$

since we have that $\delta^1_0 = 0$, $\delta^1_1 = 1$, $\delta^1_2 = 0$ and $\delta^1_3 = 0$. We now observe that, from the values of the components of the connection $\Gamma^\alpha{}_{\mu\nu}$ that were given in Table 1 of the text, we can get the values of the elements that appear in the contractions $\Gamma^\nu{}_{\nu 0}$, $\Gamma^\nu{}_{\nu 1}$, $\Gamma^\nu{}_{\nu 2}$ and $\Gamma^\nu{}_{\nu 3}$, and in this way we get for these contractions

$$\begin{aligned}
\Gamma^\nu{}_{\nu 0} &= \Gamma^0{}_{00} + \Gamma^1{}_{10} + \Gamma^2{}_{20} + \Gamma^3{}_{30} \\
&= 0 + 0 + 0 + 0 \\
&= 0, \\
\Gamma^\nu{}_{\nu 1} &= \Gamma^0{}_{01} + \Gamma^1{}_{11} + \Gamma^2{}_{21} + \Gamma^3{}_{31} \\
&= \nu'(r) + \lambda'(r) + \frac{1}{r} + \frac{1}{r} \\
&= \nu'(r) + \lambda'(r) + \frac{2}{r}, \\
\Gamma^\nu{}_{\nu 2} &= \Gamma^0{}_{02} + \Gamma^1{}_{12} + \Gamma^2{}_{22} + \Gamma^3{}_{32} \\
&= 0 + 0 + 0 + \cot(\theta)
\end{aligned}$$

$$
\begin{aligned}
&= \cot(\theta),\\
\Gamma^{\nu}{}_{\nu 3} &= \Gamma^{0}{}_{03} + \Gamma^{1}{}_{13} + \Gamma^{2}{}_{23} + \Gamma^{3}{}_{33}\\
&= 0 + 0 + 0 + 0\\
&= 0. \qquad (117)
\end{aligned}
$$

We therefore have for our contracted covariant derivative

$$
\begin{aligned}
D_\nu T^{\nu}_{\mu}(r) &= \delta^1_\mu T'_1(r) - \Gamma^{0}{}_{0\mu}T_0(r) - \Gamma^{1}{}_{1\mu}T_1(r) - \Gamma^{2}{}_{2\mu}T_2(r) - \Gamma^{3}{}_{3\mu}T_3(r) +\\
&\quad + \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\delta^1_\mu T_1(r) + \cot(\theta)\delta^2_\mu T_2(r)\\
&= \delta^1_\mu T'_1(r) - \Gamma^{0}{}_{0\mu}T_0(r) - \left\{\Gamma^{1}{}_{1\mu} - \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\delta^1_\mu\right\} T_1(r) +\\
&\quad - \left[\Gamma^{2}{}_{2\mu} - \cot(\theta)\delta^2_\mu\right] T_2(r) - \Gamma^{3}{}_{3\mu}T_3(r). \qquad (118)
\end{aligned}
$$

This must be zero for all values of μ, and therefore we must write out each one of the cases, using once more the values of the components of the connection $\Gamma^{\alpha}{}_{\mu\nu}$, shown in Table 1 of the text,

$$
\begin{aligned}
D_\nu T^{\nu}_{0}(r) &= \delta^1_0 T'_1(r) - \Gamma^{0}{}_{00}T_0(r) - \left\{\Gamma^{1}{}_{10} - \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\delta^1_0\right\} T_1(r) +\\
&\quad - \left[\Gamma^{2}{}_{20} - \cot(\theta)\delta^2_0\right] T_2(r) - \Gamma^{3}{}_{30}T_3(r)\\
&= -\Gamma^{0}{}_{00}T_0(r) - \Gamma^{1}{}_{10}T_1(r) - \Gamma^{2}{}_{20}T_2(r) - \Gamma^{3}{}_{30}T_3(r)\\
&= -0 \times T_0(r) - 0 \times T_1(r) - 0 \times T_2(r) - 0 \times T_3(r)\\
&= 0,\\
D_\nu T^{\nu}_{1}(r) &= \delta^1_1 T'_1(r) - \Gamma^{0}{}_{01}T_0(r) - \left\{\Gamma^{1}{}_{11} - \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\delta^1_1\right\} T_1(r) +\\
&\quad - \left[\Gamma^{2}{}_{21} - \cot(\theta)\delta^2_1\right] T_2(r) - \Gamma^{3}{}_{31}T_3(r)\\
&= T'_1(r) - \Gamma^{0}{}_{01}T_0(r) - \left\{\Gamma^{1}{}_{11} - \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\right\} T_1(r) +\\
&\quad -\Gamma^{2}{}_{21}T_2(r) - \Gamma^{3}{}_{31}T_3(r)\\
&= T'_1(r) - \nu'(r)T_0(r) + \left[\nu'(r) + \frac{2}{r}\right] T_1(r) - \frac{T_2(r)}{r} - \frac{T_3(r)}{r}\\
&= T'_1(r) - \nu'(r)[T_0(r) - T_1(r)] + \frac{[2T_1(r) - T_2(r) - T_3(r)]}{r},\\
D_\nu T^{\nu}_{2}(r) &= \delta^1_2 T'_1(r) - \Gamma^{0}{}_{02}T_0(r) - \left\{\Gamma^{1}{}_{12} - \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\delta^1_2\right\} T_1(r) +\\
&\quad - \left[\Gamma^{2}{}_{22} - \cot(\theta)\delta^2_2\right] T_2(r) - \Gamma^{3}{}_{32}T_3(r)\\
&= -\Gamma^{0}{}_{02}T_0(r) - \Gamma^{1}{}_{12}T_1(r) - \left[\Gamma^{2}{}_{22} - \cot(\theta)\right] T_2(r) - \Gamma^{3}{}_{32}T_3(r)\\
&= -0 \times T_0(r) - 0 \times T_1(r) + \cot(\theta)T_2(r) - \cot(\theta)T_3(r)\\
&= 0,\\
D_\nu T^{\nu}_{3}(r) &= \delta^1_3 T'_1(r) - \Gamma^{0}{}_{03}T_0(r) - \left\{\Gamma^{1}{}_{13} - \left[\nu'(r) + \lambda'(r) + \frac{2}{r}\right]\delta^1_3\right\} T_1(r) +\\
&\quad - \left[\Gamma^{2}{}_{23} - \cot(\theta)\delta^2_3\right] T_2(r) - \Gamma^{3}{}_{33}T_3(r)\\
&= -\Gamma^{0}{}_{03}T_0(r) - \Gamma^{1}{}_{13}T_1(r) - \Gamma^{2}{}_{23}T_2(r) - \Gamma^{3}{}_{33}T_3(r)\\
&= -0 \times T_0(r) - 0 \times T_1(r) - 0 \times T_2(r) - 0 \times T_3(r)\\
&= 0, \qquad (119)
\end{aligned}
$$

where we used the fact that $T_2(r) = T_3(r)$ in the third equation above. Thus we see that three of the four consistency conditions $D_\nu T_\mu^\nu(r) = 0$, those for $\mu = 0$, $\mu = 2$ and $\mu = 3$, are automatically satisfied. The only non-trivial condition is that given by $D_\nu T_1^\nu(r) = 0$, which results in

$$T_1'(r) - \nu'(r)[T_0(r) - T_1(r)] + \frac{[2T_1(r) - T_2(r) - T_3(r)]}{r} = 0, \tag{120}$$

and which can also be written as

$$\left[r\nu'(r)\right] [T_0(r) - T_1(r)] = \left[rT_1'(r)\right] + [2T_1(r) - T_2(r) - T_3(r)]. \tag{121}$$

This condition, as a condition on T_μ^ν, is required to be satisfied if this energy-momentum tensor is to be used in the right-hand side of the Einstein gravitational field equation.

A.2 Equation for $\beta(r)$

Here we present a few detailed calculations involved in the derivation of the differential equation which determined the function $\beta(\xi)$,

A.2.1 Derivation from the Field Equation

Here we calculate in terms of $\beta(r)$ the quantity $r\,[r\nu'(r)]'$. If we simply differentiate the quantity $[r\nu'(r)]$ given in Equation (62) of the text, we get

$$\begin{aligned}
r\left[r\nu'(r)\right]' &= r\,\frac{r_M}{2}\left[\frac{\beta(r) + \omega\,[r\beta'(r)]}{r - r_M\beta(r)}\right]' \\
&= r\,\frac{r_M}{2}\left\{\frac{\beta'(r) + \omega\,[r\beta'(r)]'}{r - r_M\beta(r)} - \frac{\beta(r) + \omega\,[r\beta'(r)]}{[r - r_M\beta(r)]^2}\left[1 - r_M\beta'(r)\right]\right\} \\
&= \frac{r_M}{2}\left(\frac{[r\beta'(r)] + \omega\left\{r\,[r\beta'(r)]'\right\}}{[r - r_M\beta(r)]^2}\,[r - r_M\beta(r)] + \right. \\
&\qquad \left. - \frac{\beta(r) + \omega\,[r\beta'(r)]}{[r - r_M\beta(r)]^2}\left\{r - r_M\left[r\beta'(r)\right]\right\}\right) \\
&= \frac{r_M}{2[r - r_M\beta(r)]^2}\times \\
&\quad \times\left[\left(\left[r\beta'(r)\right] + \omega\left\{r\left[r\beta'(r)\right]'\right\}\right)[r - r_M\beta(r)] + \right. \\
&\qquad \left. - \left\{\beta(r) + \omega\left[r\beta'(r)\right]\right\}\left\{r - r_M\left[r\beta'(r)\right]\right\}\right] \\
&= \frac{r_M}{2[r - r_M\beta(r)]^2}\times \\
&\quad \times\left(\omega\,[r - r_M\beta(r)]\left\{r\left[r\beta'(r)\right]'\right\} + r\left[r\beta'(r)\right] - r_M\beta(r)\left[r\beta'(r)\right] + \right. \\
&\qquad \left. - r\beta(r) + r_M\beta(r)\left[r\beta'(r)\right] - \omega r\left[r\beta'(r)\right] + \omega r_M\left[r\beta'(r)\right]^2\right) \\
&= \frac{r_M}{2[r - r_M\beta(r)]^2}\left(\omega\,[r - r_M\beta(r)]\left\{r\left[r\beta'(r)\right]'\right\} + \omega r_M\left[r\beta'(r)\right]^2 + \right. \\
&\qquad \left. + (1 - \omega) r\left[r\beta'(r)\right] - r\beta(r)\right),
\end{aligned} \tag{122}$$

so that we have

$$r\left[r\nu'(r)\right]' = r_M \frac{\left(\begin{array}{c}\omega\left[r - r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\} + \omega r_M\left[r\beta'(r)\right]^2 + \\ +(1-\omega)r\left[r\beta'(r)\right] - r\beta(r)\end{array}\right)}{2[r - r_M\beta(r)]^2}, \tag{123}$$

which therefore determines $r\left[r\nu'(r)\right]'$, and indirectly also determines $\nu''(r)$, in terms of $\beta(r)$.

A.2.2 Simplification to Final Form

Here we will work on Equation (58) of the text, the one that will become the equation that determines $\beta(r)$, in order to put it in final form for further analysis. That equation is written as

$$r\left[r\nu'(r)\right]' + \left[r\nu'(r)\right]^2 - \left[r\lambda'(r)\right]\left[r\nu'(r)\right] = \left[r\lambda'(r)\right] + \mathrm{e}^{2\lambda(r)}\omega\mathfrak{T}(r). \tag{124}$$

We will now work, in turn, on the right-hand and left-hand sides of this equation. Using the results obtained in the text for $\lambda(r)$, $\lambda'(r)$ and $\mathfrak{T}(r)$, we can write the right-hand side as

$$\begin{aligned}
\text{RHS} &= -\frac{r_M}{2}\frac{\beta(r) - [r\beta'(r)]}{r - r_M\beta(r)} + \frac{r\omega r_M\beta'(r)}{r - r_M\beta(r)} \\
&= \frac{r_M}{2}\frac{2\omega\left[r\beta'(r)\right]}{r - r_M\beta(r)} - \frac{r_M}{2}\frac{\beta(r) - [r\beta'(r)]}{r - r_M\beta(r)} \\
&= \frac{r_M}{2}\frac{2\omega\left[r\beta'(r)\right] - \beta(r) + [r\beta'(r)]}{r - r_M\beta(r)} \\
&= \frac{r_M}{2[r - r_M\beta(r)]}\left\{(1+2\omega)\left[r\beta'(r)\right] - \beta(r)\right\}.
\end{aligned} \tag{125}$$

For convenience in the manipulations that follow afterward, we may write the final form of this equation as

$$\begin{aligned}
\text{RHS} &= \frac{r_M}{4[r - r_M\beta(r)]^2}\left\{2(1+2\omega)\left[r\beta'(r)\right] - 2\beta(r)\right\}\left[r - r_M\beta(r)\right] \\
&= \frac{r_M}{4[r - r_M\beta(r)]^2}\left\{2(1+2\omega)r\left[r\beta'(r)\right] - 2r\beta(r) + \right. \\
&\qquad \left. -2(1+2\omega)r_M\beta(r)\left[r\beta'(r)\right] + 2r_M\beta^2(r)\right\}.
\end{aligned} \tag{126}$$

As one can see, this involves the function $\beta(r)$, its derivative $\beta'(r)$, and the parameters ω and r_M. Using again the results obtained in the text for the relevant quantities, we can now write the left-hand side of our θ component equation as

$$\begin{aligned}
\text{LHS} &= \frac{r_M}{2[r - r_M\beta(r)]^2}\left(\omega\left[r - r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\} + \right. \\
&\qquad +\omega r_M\left[r\beta'(r)\right]^2 + \\
&\qquad \left. +(1-\omega)r\left[r\beta'(r)\right] - r\beta(r)\right) + \\
&\quad + \frac{r_M^2}{4[r - r_M\beta(r)]^2}\left\{\beta(r) + \omega\left[r\beta'(r)\right]\right\}^2 + \\
&\quad + \frac{r_M^2}{4[r - r_M\beta(r)]^2}\left\{\beta(r) - \left[r\beta'(r)\right]\right\}\left\{\beta(r) + \omega\left[r\beta'(r)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{r_M}{4\left[r-r_M\beta(r)\right]^2}\times \\
&\times\Bigg(2\omega\left[r-r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\}+2\omega r_M\left[r\beta'(r)\right]^2+ \\
&\quad +2(1-\omega)r\left[r\beta'(r)\right]-2r\beta(r)+r_M\beta^2(r)+ \\
&\quad +2\omega r_M\beta(r)\left[r\beta'(r)\right]+\omega^2 r_M\left[r\beta'(r)\right]^2+r_M\beta^2(r)+ \\
&\quad +\omega r_M\beta(r)\left[r\beta'(r)\right]-r_M\beta(r)\left[r\beta'(r)\right]-\omega r_M\left[r\beta'(r)\right]^2\Bigg).
\end{aligned}
\tag{127}
$$

Some simplifications can now be made, so that we may write that

$$
\begin{aligned}
\mathrm{LHS} &= \frac{r_M}{4\left[r-r_M\beta(r)\right]^2}\Bigg(2\omega\left[r-r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\}+ \\
&\quad +\omega(1+\omega)r_M\left[r\beta'(r)\right]^2+ \\
&\quad +2(1-\omega)r\left[r\beta'(r)\right]-2r\beta(r)+ \\
&\quad +2r_M\beta^2(r)+(3\omega-1)r_M\beta(r)\left[r\beta'(r)\right]\Bigg).
\end{aligned}
\tag{128}
$$

Using the expressions for the left-hand and right-hand sides we may now write for the θ component of the field equation,

$$
\begin{aligned}
&2\omega\left[r-r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\}+\omega(1+\omega)r_M\left[r\beta'(r)\right]^2+ \\
&\quad +2(1-\omega)r\left[r\beta'(r)\right]-2r\beta(r)+2r_M\beta^2(r)+(3\omega-1)r_M\beta(r)\left[r\beta'(r)\right] \\
&\qquad = 2(1+2\omega)r\left[r\beta'(r)\right]-2r\beta(r)+ \\
&\qquad\quad -2(1+2\omega)r_M\beta(r)\left[r\beta'(r)\right]+2r_M\beta^2(r).
\end{aligned}
\tag{129}
$$

Some of the terms now cancel off, and passing all the remaining terms to the left-hand side we are left with

$$
\begin{aligned}
&2\omega\left[r-r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\}+\omega(1+\omega)r_M\left[r\beta'(r)\right]^2+ \\
&\quad +(2-2\omega)r\left[r\beta'(r)\right]+(3\omega-1)r_M\beta(r)\left[r\beta'(r)\right]+ \\
&\quad -(2+4\omega)r\left[r\beta'(r)\right]+(2+4\omega)r_M\beta(r)\left[r\beta'(r)\right] = 0.
\end{aligned}
\tag{130}
$$

One can see that some more terms will cancel off. For convenience we will write this equation with an extra overall factor of r_M, as

$$
\begin{aligned}
&2\omega r_M\left[r-r_M\beta(r)\right]\left\{r\left[r\beta'(r)\right]'\right\}+\omega(1+\omega)r_M^2\left[r\beta'(r)\right]^2+ \\
&\quad -6\omega r r_M\left[r\beta'(r)\right]+(1+7\omega)r_M^2\beta(r)\left[r\beta'(r)\right] = 0,
\end{aligned}
\tag{131}
$$

so that it is now ready for the further manipulations made in the text.

A.2.3 Derivation from the Consistency Condition

Here we derive Equation (74) of the text from the consistency condition shown in Equation (27) of the text. If we use the results obtained in the text for $\nu'(r)$ and $\mathfrak{T}(r)$ the consistency condition can be written as

$$\frac{r_M}{2}\,\frac{\beta(r)+\omega\,[r\beta'(r)]}{r-r_M\beta(r)}=\frac{2\omega}{1+\omega}-\frac{\omega}{1+\omega}\,\frac{[r\beta''(r)]}{\beta'(r)}. \tag{132}$$

If we now use the fact that

$$r\beta''(r)=\left[r\beta'(r)\right]'-\beta'(r), \tag{133}$$

we can write this equation as

$$\begin{aligned}
\frac{r_M}{2}\,\frac{\beta(r)+\omega\,[r\beta'(r)]}{r-r_M\beta(r)} &= \frac{2\omega}{1+\omega}-\frac{\omega}{1+\omega}\,\frac{\left\{[r\beta'(r)]'-\beta'(r)\right\}}{\beta'(r)}\\
&= \frac{2\omega}{1+\omega}-\frac{\omega}{1+\omega}\,\frac{[r\beta'(r)]'}{\beta'(r)}+\frac{\omega}{1+\omega}\\
&= \frac{3\omega}{1+\omega}-\frac{\omega}{1+\omega}\,\frac{[r\beta'(r)]'}{\beta'(r)}.
\end{aligned} \tag{134}$$

Making the crossed-products in order to eliminate the denominators we get

$$\begin{aligned}
&(1+\omega)r_M\beta(r)\beta'(r)+\omega(1+\omega)r_M\beta'(r)\left[r\beta'(r)\right]\\
&\quad= [r-r_M\beta(r)]\left\{6\omega\beta'(r)-2\omega\left[r\beta'(r)\right]'\right\}\\
&\quad= 6\omega\left[r\beta'(r)\right]-2\omega\left\{r\left[r\beta'(r)\right]'\right\}-6\omega r_M\beta(r)\beta'(r)+2\omega r_M\beta(r)\left[r\beta'(r)\right]'.
\end{aligned} \tag{135}$$

Making the product by an extra factor of r and reorganizing the terms we get

$$\begin{aligned}
&(1+\omega)r_M\beta(r)\left[r\beta'(r)\right]+\omega(1+\omega)r_M\left[r\beta'(r)\right]^2\\
&\quad= 6\omega r\left[r\beta'(r)\right]-2\omega r\left\{r\left[r\beta'(r)\right]'\right\}-6\omega r_M\beta(r)\left[r\beta'(r)\right]+2\omega r_M\beta(r)\left\{r\left[r\beta'(r)\right]'\right\}\\
&\quad= -2\omega\,[r-r_M\beta(r)]\left\{r\left[r\beta'(r)\right]'\right\}+6\omega r\left[r\beta'(r)\right]-6\omega r_M\beta(r)\left[r\beta'(r)\right],
\end{aligned} \tag{136}$$

so that, passing all terms to the left-hand side and finally multiplying by an extra factor of r_M, we get

$$\begin{aligned}
2\omega r_M\,[r-r_M\beta(r)]\left\{r\left[r\beta'(r)\right]'\right\}+\omega(1+\omega)r_M^2\left[r\beta'(r)\right]^2+\\
-6\omega r r_M\left[r\beta'(r)\right]+(1+7\omega)r_M^2\beta(r)\left[r\beta'(r)\right] &= 0.
\end{aligned} \tag{137}$$

One can now see that this is exactly the same expression shown in Equation (74) of the text.