Complex Analysis of Real Functions

VII: A Simple Extension of the Cauchy-Goursat Theorem

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August 19, 2018

Abstract

In the context of the complex-analytic structure within the open unit disk, that was established in a previous paper, here we establish a simple generalization of the Cauchy-Goursat theorem of complex analytic functions. We do this first for the case of inner analytic functions, and then generalize the result to all analytic functions. We thus show that the Cauchy-Goursat theorem holds even if the complex function has isolated singularities located *on* the integration contour, so long as these are all *integrable* ones.

1 Introduction

In previous papers [1–6] we have shown that there is a correspondence between, on the one hand, real functions and other real objects on the unit circle, and on the other hand, inner analytic functions within the open unit disk of the complex plane [7]. This correspondence is based on the complex-analytic structure which we introduced in [1]. That complex-analytic structure includes the concept of inner analytic functions, two analytic operations on them, which we named angular differentiation and angular integration, and a scheme for the classification of all the possible singularities of these functions.

This classification scheme separates the singularities as either soft or hard ones, depending on whether or not the limit of the function to the singular point exists. As part of that classification scheme we also introduced gradations of both hardness and softness for the singularities, given by integers degrees. In particular, a hard singularity which becomes soft under a single angular integration of the inner analytic function is a borderline hard one, with degree of hardness zero. In a previous paper [1] we have shown that both soft and borderline hard singularities are integrable ones, while the hard singularities with strictly positive degrees of hardness are non-integrable ones.

Here we will show that one can extend the Cauchy-Goursat theorem, by weakening its hypotheses, so that both soft and borderline hard isolated singularities are allowed on the integration contour. We will prove this first for inner analytic functions, and then generalize the result to arbitrary complex analytic functions, using conformal transformations.

For ease of reference, we include here a one-page synopsis of the complex-analytic structure introduced in [1]. It consists of certain elements within complex analysis [7], as well as of their main properties.

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Synopsis: The Complex-Analytic Structure

An *inner analytic function* $w(z)$ is simply a complex function which is analytic within the open unit disk. An inner analytic function that has the additional property that $w(0) = 0$ is a *proper inner analytic function.* The *angular derivative* of an inner analytic function is defined by

$$w^{\cdot}(z) = \imath z \, \frac{dw(z)}{dz}. \tag{1}$$

By construction we have that $w^{\cdot}(0) = 0$, for all $w(z)$. The *angular primitive* of an inner analytic function is defined by

$$w^{-1\cdot}(z) = -\imath \int_{0}^{z} dz' \, \frac{w(z') - w(0)}{z'}. \tag{2}$$

By construction we have that $w^{-1\cdot}(0) = 0$, for all $w(z)$. In terms of a system of polar coordinates (ρ, θ) on the complex plane, these two analytic operations are equivalent to differentiation and integration with respect to θ, taken at constant ρ. These two operations stay within the space of inner analytic functions, they also stay within the space of proper inner analytic functions, and they are the inverses of one another. Using these operations, and starting from any proper inner analytic function $w^{0\cdot}(z)$, one constructs an infinite *integral-differential chain* of proper inner analytic functions,

$$\left\{ \ldots, w^{-3\cdot}(z), w^{-2\cdot}(z), w^{-1\cdot}(z), w^{0\cdot}(z), w^{1\cdot}(z), w^{2\cdot}(z), w^{3\cdot}(z), \ldots \right\}. \tag{3}$$

Two different such integral-differential chains cannot ever intersect each other. There is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, namely the null chain, in which all members are the null function $w(z) \equiv 0$.

A general scheme for the classification of all possible singularities of inner analytic functions is established. A singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity.* Angular integration takes soft singularities to other soft singularities, and angular differentiation takes hard singularities to other hard singularities.

Gradations of softness and hardness are then established. A hard singularity that becomes a soft one by means of a single angular integration is a *borderline hard* singularity, with degree of hardness zero. The *degree of softness* of a soft singularity is the number of angular differentiations that result in a borderline hard singularity, and the *degree of hardness* of a hard singularity is the number of angular integrations that result in a borderline hard singularity. Singularities which are either soft or borderline hard are integrable ones. Hard singularities which are not borderline hard are non-integrable ones.

Given an integrable real function $f(\theta)$ on the unit circle, one can construct from it a unique corresponding inner analytic function $w(z)$. Real functions are obtained through the $\rho \to 1_{(-)}$ limit of the real and imaginary parts of each such inner analytic function and, in particular, the real function $f(\theta)$ is obtained from the real part of $w(z)$ in this limit. The pair of real functions obtained from the real and imaginary parts of one and the same inner analytic function are said to be mutually Fourier-conjugate real functions.

Singularities of real functions can be classified in a way which is analogous to the corresponding complex classification. Integrable real functions are typically associated with inner analytic functions that have singularities which are either soft or at most borderline hard. This ends our synopsis.

Unlike what was the case for the previous papers in this series, which were focused mostly on the role played by this complex-analytic structure in the analysis of real functions and other real objects, the result presented in this paper concerns directly the theory of complex analytic functions. Before we tackle the central issue of this paper, we must present a review and refinement of some previous results, which were part of that complex-analytic structure.

For the work to be developed in this paper it is important to recall that, as was already noted in [1], whenever the singularities on the unit circle are branch points, the corresponding branch cuts are to be extended *outward* from the unit circle, so that there are no branch cuts crossing the unit disk. In fact, in order to simplify the arguments to be developed here, we adopt this recipe as an integral part of the complex-analytic structure.

Some of the material contained in this paper can be seen as a development, reorganization and extension of part of the material found, sometimes still in rather rudimentary form, in the papers [8–12].

2 Refinement of Two Previous Results

The discussions of two of the results related to the complex-analytic structure introduced in [1], whose proofs were presented in that paper, turn out to be somewhat incomplete for our needs here, so that the proofs presented there must be somewhat refined. These are the discussions regarding the fact that soft singularities must be integrable ones (Property 5.1 of Definition 5), and the fact that borderline hard singularities must be integrable ones (Property 8.1 of Definition 8). Let us discuss and refine each one in turn.

With regard to the fact that a soft singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle must be an integrable one, the discussion given in [1] takes us to the point where it is shown that the integral of $w(z)$ exists on all simple curves contained within the unit disk that connect to the point z_1. However, we neglected to point out that certain integrals involving $w(z)$ have all the same *value*, which is equivalent to the fact that the angular primitive $w^{-1\cdot}(z)$ is well-defined at z_1. Let us repeat the argument here. Therefore, we now review the following important property of soft singularities, first established in [1].

Property 5.1: *A soft singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle is necessarily an integrable one.*

This is so because the angular integration of $w(z)$ produces its angular primitive, an inner analytic function $w^{-1\cdot}(z)$ which also has at z_1 a soft singularity, and therefore is well defined at that point. Since the value of $w^{-1\cdot}(z)$ at z_1 is given by an integral involving $w(z)$ along a curve from the origin to z_1, as shown in Equation (2), that integral must therefore exist and result in a finite complex number, for all curves within the open unit disk that go from 0 to z_1. Since the other factor involved in the integrand of that integral is a regular function which is different from zero in a neighborhood around z_1, say an open disk of radius ε as shown in Figure 1, this implies that $w(z)$ must be integrable around z_1. Therefore, the singularity of $w(z)$ at z_1 must be an integrable one.

In addition to this, since $w^{-1\cdot}(z)$ has a definite finite complex value at z_1, we may also conclude that the value of the integral giving it does not depend on the integration contour from 0 to z_1, that is, on the direction along which that curve connects to z_1. Then the Cauchy-Goursat theorem in its usual form, which implies that integrals from the origin to any point z_0 within the open unit disk are independent of the integration contours connecting those two points and contained within the open unit disk, allows us to generalize

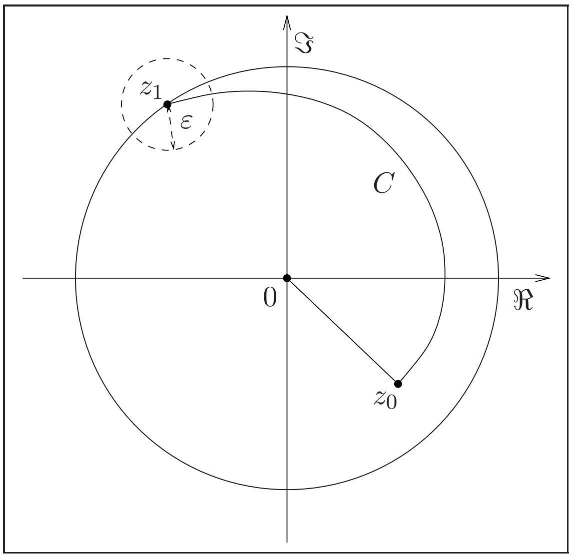

Figure 1: The unit circle of the complex plane, the singular point z_1, the origin 0, the internal point z_0, the neighborhood of radius ε, the integration contour C and the straight segment connecting z_0 to the origin.

the result, in a straightforward way, from curves starting at the point 0 to curves starting at any internal point z_0 within the open unit disk, by simply connecting the origin and z_0 by means of any simple curve, for example the straight segment shown in Figure 1. We therefore conclude that, given an integration contour C contained within the unit disk and going from z_0 to z_1, the integral

$$\int_C dz\,\frac{w(z) - w(0)}{z} \tag{4}$$

does not depend on the contour. We thus establish this property in a more complete way.

With regard to the fact that a borderline hard singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle must be an integrable one, we again neglected to point out in [1] that the same set of integrals from z_0 to z_1 discussed in the previous case, along any simple curve contained within the unit disk and connecting those two points, are all equal, which once more is equivalent to the fact that the angular primitive $w^{-1\bullet}(z)$ is well-defined at z_1. Let us repeat the argument here. Therefore, we now review the following important property of borderline hard singularities, first established in [1].

Property 8.1: *A borderline hard singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle must be an integrable one.*

This is so because the angular integration of $w(z)$ produces its angular primitive, an inner analytic function $w^{-1\bullet}(z)$ which has at z_1 a soft singularity, given that the singularity of $w(z)$ at that point is a borderline hard one, and therefore $w^{-1\bullet}(z)$ is well defined at the point z_1. Since the value of $w^{-1\bullet}(z)$ at z_1 is given by an integral involving $w(z)$ along a curve from the origin to z_1, as shown in Equation (2), that integral must therefore exist and result in a finite complex number, for all curves within the unit disk that go from 0 to z_1. Since the other factor involved in the integrand of that integral is a regular function which is different from zero in the neighborhood around z_1, this implies that $w(z)$ must be integrable around z_1. Therefore, the singularity of $w(z)$ at z_1 must be an integrable one.

In addition to this, since $w^{-1\bullet}(z)$ has a definite finite complex value at z_1, we may also conclude that the value of the integral giving it does not depend on the integration contour from the origin to z_1, that is, on the direction along which that curve connects to z_1. Just as was noted before in the discussion of the previous case, at this point the Cauchy-Goursat theorem allows us to generalize the result, in a straightforward way, from curves starting at the point 0 to curves starting at any internal point z_0 on the open unit disk. We therefore conclude that, given an integration contour C contained within the unit disk and going from z_0 to z_1, the integral

$$\int_C dz\,\frac{w(z)-w(0)}{z} \tag{5}$$

does not depend on the contour. We thus establish this property in a more complete way.

We may express these two results, about the singularities being integrable, in a more concise way, by simply stating that what we mean by the integrability of an inner analytic function $w(z)$ around a singular point z_1 on the unit circle is that the integrals shown in Equation (2) on curves contained within the open unit disk and going from any internal point z_0 to that singular point both exist and are independent of the curves. This is, of course, equivalent to the statement that the angular primitive of $w(z)$ exists and is well-defined at z_1.

Once again we recall that, as was noted in [1], whenever the singularities on the unit circle are branch points, it is understood that the corresponding branch cuts are to be extended *outward* from the unit circle, so that no branch cuts cross the unit disk. In this way there can be no crossings between branch cuts and integration contours within the open unit disk. This simplifies the arguments, since such crossings would force us to consider the fact that the integration contours might be changing from one leaf of a Riemann surface to another. However, the fact that this is *not* an essential hypothesis is apparent when one considers that for *closed* integration contours these crossings would necessarily happen in pairs, each pair representing a change of leafs followed by a change back to the original leaf, given that all branch cuts within the open unit disk must cross it completely, since there are no singularities of $w(z)$ within the open unit disk.

3 Extension of the Cauchy-Goursat Theorem

In either one of the two situations examined in Section 2, in which the inner analytic function $w(z)$ has either a soft singularity or a borderline hard singularity at a point z_1 on the unit circle, we discovered that, given an integration contour C contained within the unit disk and going from any internal point z_0 to z_1, the integral

$$\int_C dz\,\frac{w(z)-w(0)}{z} \tag{6}$$

does not depend on the contour. We now observe that, if $w(z)$ is any inner analytic function, then $w_p(z) = w(z) - w(0)$ is a *proper* inner analytic function, since $w_p(0) = 0$. Therefore what we have here is the statement that the integral

$$\int_C dz\,\frac{w_p(z)}{z} \tag{7}$$

does not depend on the contour, for all integration contours C contained within the unit disk that go from z_0 to z_1, and for all proper inner analytic functions within the unit disk that have an integrable singularity at z_1. Let us now consider the integral

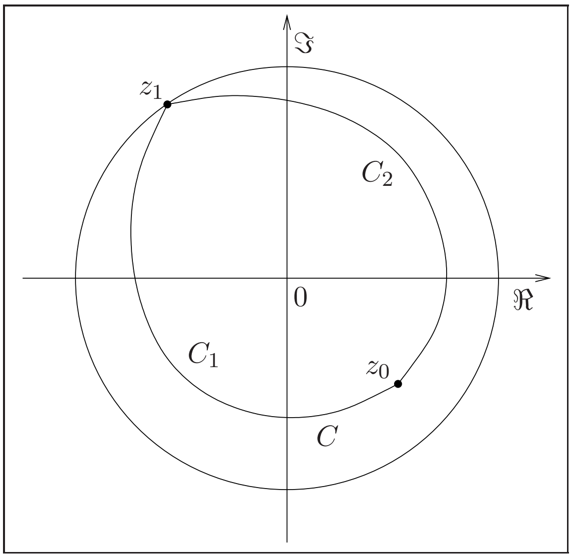

Figure 2: The unit circle of the complex plane, the singular point z_1, the internal point z_0, and the integration contours involved in the proof of the extended Cauchy-Goursat theorem for inner analytic function within the unit disk.

$$\int_C dz\, w(z), \tag{8}$$

for an arbitrary inner analytic function $w(z)$ that has an integrable singularity at z_1. Since $w(z)$ is necessarily regular at $z = 0$, it follows that the function $w_p(z) = zw(z)$ is a *proper* inner analytic function, given that $w_p(0) = 0$. In addition to this, since the function z is analytic everywhere, it also follows that $w_p(z)$ and $w(z)$ have the same singularity structure, and thus we can write

$$\int_C dz\, w(z) = \int_C dz\, \frac{w_p(z)}{z}, \tag{9}$$

where by the statement involving Equation (7) this last integral is independent of the integration contour C. Therefore, we have the statement that

$$\int_C dz\, w(z) \tag{10}$$

is independent of the contour, for all integration contours C within the unit disk that go from z_0 to z_1, and for all inner analytic functions that have an integrable singularity at z_1.

We have thus determined that in these two cases the integral of an arbitrary inner analytic function $w(z)$ from an arbitrary point z_0, internal to the open unit disk, to the point z_1 on the unit circle, where in either case $w(z)$ has an isolated integrable singularity, along an integration contour contained within the closed unit disk and that touches the unit circle only at z_1, is independent of that integration contour from z_0 to z_1. Therefore, given two different such curves, such as the curves C_1 and C_2 illustrated in Figure 2, we may immediately conclude that the integral of $w(z)$ over the *closed* integration contour C formed by the two curves is zero,

$$\oint_C dz\, w(z) = 0. \tag{11}$$

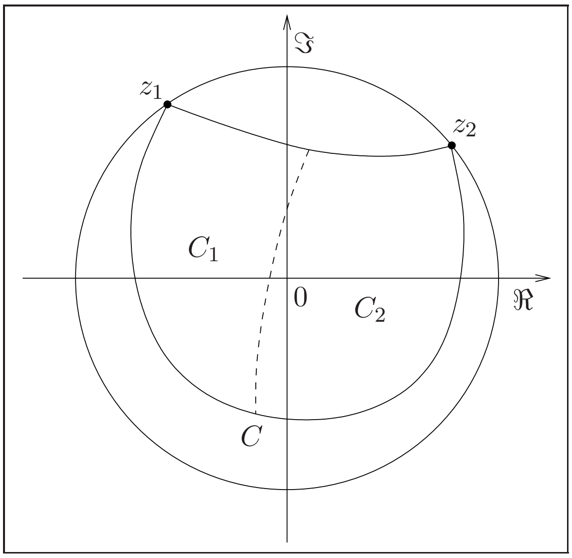

Figure 3: The unit circle of the complex plane, the integration contour C (solid line) with two singular points z_1 and z_2, showing how it can be decomposed into two integration contours C_1 and C_2, each one with only one singular point, by a simple cut (dashed line).

Since z_0 is an arbitrary internal point, this is valid for all closed simple curves C within the unit disk, that touch the unit circle only at z_1. Observe that what we have concluded here is, in fact, that the validity of the Cauchy-Goursat theorem, for the case of inner analytic functions, is not disturbed by the presence of an isolated singularity *on* the integration contour, so long as this singularity is either a *soft* one or a *borderline hard* one or, in other words, so long as singularities like this are all *integrable* ones. It is quite clear, therefore, that this result constitutes an extension of the usual form of the Cauchy-Goursat theorem, one which is valid at least for inner analytic functions within the unit disk.

This result for a single point of singularity can then be trivially extended, by contour manipulation and the repeated use of the Cauchy-Goursat theorem in its usual form, to integration contours that touch the unit circle on a finite number of points, at all of which $w(z)$ has isolated integrable singularities. Therefore, as a side effect of the arguments presented in Section 2, in this section we will prove the following theorem.

Theorem 1: *Given an inner analytic function $w(z)$, and a closed integration contour C contained within the closed unit disk, that touches the unit circle only at a set of points satisfying one of two conditions, either points where $w(z)$ is analytic, or points where $w(z)$ has isolated integrable singularities, of which there must be a finite number, it follows that the integral of $w(z)$ over the contour C is zero,*

$$\oint_C dz\, w(z) = 0. \tag{12}$$

Proof 1.1:

We start with contours that touch the unit circle at a single integrable singular point z_1, such as the one shown in Figure 2, and by just using the results of Section 2, as expressed by Equation (11), to simply state that the integral of $w(z)$ over any such contour is zero. Next, given a closed contour C that touches the unit circle at two separate singular points z_1 and z_2, such as the one shown in Figure 3, it can always be separated into two closed

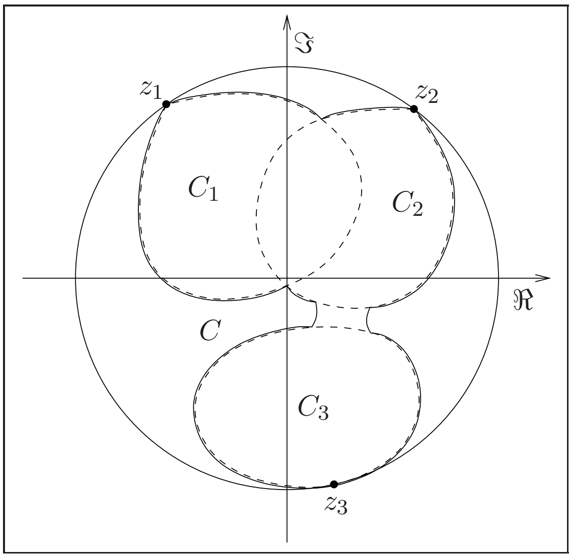

Figure 4: The unit circle of the complex plane, the singular points z_1, z_2 and z_3, and the corresponding integration contours C_1, C_2 and C_3 (dashed lines), joined into a single overall contour C (solid line).

contours C_1 and C_2, each one of which touches the unit circle at only one singular point, as in the example shown in Figure 3, by a simple cut (dashed line). When the two separate closed contours C_1 and C_2 are joined together to form the complete contour C, due to the orientation of the contours the integrals over the cut, which is traversed twice, once in each direction, cancel out. This can also be done for a contour that touches the unit circle at any finite number of separate singular points.

Another way to think about this is to consider that one can construct any contour such as those described in the statement of Theorem 1 by joining together a finite number of contours, each one of which touches the unit circle at only one singular point, as is illustrated for the case of three contours in Figure 4. Thus we see that the enormous freedom to deform integration contours within the open unit disk without changing the value of the integrals, which is given to us by the Cauchy-Goursat theorem in its usual form, can be used to reduce a generic closed contour, that touches N isolated integrable singular points on the unit circle, to a set of N closed contours, each one of which touches the unit circle at only one such singular point. This effectively reduces the proof for the large and more complex contour to that for the simple contour with only one singularity.

In addition to this, using the Cauchy-Goursat theorem in its usual form, we may also deform any contour so that it morphs into one that touches the unit circle at any points on that circle where $w(z)$ is analytic. In other words, the integration contour may also run along any parts of the unit circle on which $w(z)$ has no singularities at all. This completes, therefore, the proof of Theorem 1.

In this section we have established that the extended version of the Cauchy-Goursat theorem, allowing for the presence of a finite number of isolated integrable singularities on the integration contour, holds for all inner analytic functions within the unit disk. In Section 5 we will generalize that result to all complex analytic functions, anywhere on the complex plane, using conformal transformations. Therefore, before we discuss the generalization of the theorem we must discuss these conformal transformations.

4 Conformal Transformations and Singularities

The validity of the extended version of the Cauchy-Goursat theorem can be generalized to all complex analytic functions integrated on arbitrarily given closed integration contours, through the use of conformal transformations. In order to do this, let us first establish the definition and the notation for a conformal transformation. This is essentially a shorter version of the discussion on this topic which was given in Section 4 of a previous paper [5]. Consider therefore two complex variables z_a and z_b and the corresponding complex planes, a complex analytic function $\gamma(z)$ defined on the complex plane z_a with values on the complex plane z_b, and its inverse function, which is a complex analytic function $\gamma^{(-1)}(z)$ defined on the complex plane z_b with values on the complex plane z_a,

$$\begin{aligned} z_b &= \gamma(z_a), \\ z_a &= \gamma^{(-1)}(z_b). \end{aligned} \tag{13}$$

Let us point out here that these relations immediately imply that

$$\begin{aligned} \frac{dz_b}{dz_a} &= \frac{d\gamma(z_a)}{dz_a}, \\ \frac{dz_a}{dz_b} &= \frac{d\gamma^{(-1)}(z_b)}{dz_b}, \end{aligned} \tag{14}$$

which in turn imply that

$$\frac{d\gamma(z_a)}{dz_a}\frac{d\gamma^{(-1)}(z_b)}{dz_b} = 1, \tag{15}$$

for all pairs of points z_a and z_b related by the conformal transformation. This means that any point where the derivative of $\gamma(z_a)$ has a zero on the z_a plane corresponds to a point where the derivative of $\gamma^{(-1)}(z_b)$ has a singularity on the z_b plane, and vice-versa.

Consider a bounded and simply connected open region S_a on the complex plane z_a and its image S_b under $\gamma(z)$, which is a similar region on the complex plane z_b. It can be shown that if $\gamma(z)$ is analytic on S_a, is invertible there, and its derivative has no zeros there, then its inverse function $\gamma^{(-1)}(z)$ has these same three properties on S_b, and the mapping between the two complex planes established by $\gamma(z)$ and $\gamma^{(-1)}(z)$ is conformal, in the sense that it preserves the angles between oriented curves at points where they cross each other. The famous *Riemann mapping theorem* states that such a conformal transformation $\gamma(z)$ always exists between the open unit disk S_a and any region S_b. In addition to this, these properties of $\gamma(z)$ and $\gamma^{(-1)}(z)$ can be extended to the boundary of the regions so long as these boundaries are differentiable simple curves.

Consider therefore that the regions under consideration are the interiors of simple closed curves. One of these curves will be the unit circle C_a on the complex plane z_a, and the other will be a given closed differentiable simple curve C_b on the complex plane z_b. Since $\gamma(z_a)$, being analytic, is in particular a continuous function, the image on the z_b plane of the unit circle C_a on the z_a plane must be a continuous closed curve C_b. We can also see that C_b must be a simple curve, because the fact that $\gamma(z_a)$ is invertible on C_a means that it cannot have the same value at two different points of C_a, and therefore no two points of C_b can be the same. Consequently, the curve C_b cannot self-intersect. Finally, the fact that C_b must be a differentiable curve is a simple consequence of the facts that the $\gamma(z_a)$ transformation is conformal and that the unit circle C_a is a differentiable curve.

Given any analytic function $w_a(z_a)$ on the z_a plane, the conformal transformation $\gamma(z_a)$ maps it to a corresponding function $w_b(z_b)$ on the z_b plane, and the inverse conformal transformation $\gamma^{(-1)}(z_b)$ maps that function back to the function $w_a(z_a)$ on the z_a plane. We can do this by simply composing either $w_b(z_b)$ or $w_a(z_a)$ with either the transformation or its inverse, and simply passing the values of the functions,

$$\begin{aligned}
w_b(z_b) &= w_a(z_a) \\
&= w_a\left(\gamma^{(-1)}(z_b)\right), \\
w_a(z_a) &= w_b(z_b) \\
&= w_b(\gamma(z_a)) .
\end{aligned} \tag{16}$$

Since the composition of two analytic functions is also an analytic function, and since $\gamma(z_a)$ is analytic on the closed unit disk, whenever $w_b(z_b)$ is analytic on the z_b plane the corresponding function $w_a(z_a)$ will also be analytic at the corresponding points on the z_a plane. Of course, where one of these two functions has an isolated singularity on its plane of definition, so will the other on the corresponding point in the other plane. Note that if any of these functions is integrated over a closed integration contour on which it has any isolated integrable singularities, whenever these singularities are branch points we assume that the corresponding branch cuts extend *outward* from the integration contours.

The concepts of a soft singularity and of a borderline hard singularity can be immediately extended from the case of inner analytic functions within the unit disk to singularities of arbitrary complex analytic functions anywhere on the complex plane. The concept of a soft singularity of $w(z)$ at z_1 depends only on the existence of the $z \to z_1$ limit of $w(z)$. The concept of a hard singularity of $w(z)$ at z_1 depends only on the non-existence of that same limit. Finally, the concept of a borderline hard singularity can be defined as that of a hard singularity which is nevertheless an integrable one. In order to discuss what happens with the singularities under the conformal transformation, we must establish a few simple preliminary results, by means of the following lemmas.

Lemma 1: *If $z_{b,1}$ is a singular point of $w_b(z_b)$, then the corresponding point $z_{a,1}$ under the conformal transformation is a singular point of $w_a(z_a)$.*

Since according to Equations (16) we have that $w_b(z_{b,1}) = w_a\left(\gamma^{(-1)}(z_{b,1})\right)$ and since $\gamma^{(-1)}(z_b)$ is analytic within and on the image of the unit circle by the conformal transformation, if $w_a(z_a)$ were analytic at $z_{a,1}$, then $w_b(z_b)$ would be analytic at $z_{b,1}$, because the composition of two analytic functions is also an analytic function. Therefore, if $w_b(z_b)$ is singular at $z_{b,1}$, then $w_a(z_a)$ must be singular at $z_{a,1}$. This establishes Lemma 1.

Lemma 2: *If the singularity of $w_b(z_b)$ at $z_{b,1}$ is a soft one, then the singularity of $w_a(z_a)$ at the corresponding singular point $z_{a,1}$ under the conformal transformation is also a soft singularity.*

Since according to the definition in Equations (16) we have that $w_a(z_a) = w_b(z_b)$ and since the $z_b \to z_{b,1}$ limit on the z_b plane corresponds, through the continuous conformal transformation, to the $z_a \to z_{a,1}$ limit on the z_a plane, it follows that if the limit

$$\lim_{z_b \to z_{b,1}} w_b(z_b) \tag{17}$$

exists, then so does the limit

$$\lim_{z_a \to z_{a,1}} w_a(z_a). \tag{18}$$

Therefore, if the singularity of $w_b(z_b)$ at $z_{b,1}$ is a soft one, which means that the first limit exists, then the singularity of $w_a(z_a)$ at $z_{a,1}$ is also a soft singularity, since in this case the second limit also exists. This establishes Lemma 2.

Lemma 3: *If the singularity of $w_b(z_b)$ at $z_{b,1}$ is a hard one, then the singularity of $w_a(z_a)$ at the corresponding singular point $z_{a,1}$ under the conformal transformation is also a hard singularity.*

By an argument similar to the one used in Lemma 2, since according to the definition in Equations (16) we have that $w_a(z_a) = w_b(z_b)$, and since the $z_b \to z_{b,1}$ and $z_a \to z_{a,1}$ limits correspond to one another, if $w_b(z_b)$ is *not* well defined at $z_{b,1}$, which means that the limit

$$\lim_{z_b \to z_{b,1}} w_b(z_b) \tag{19}$$

does *not* exist, then the limit

$$\lim_{z_a \to z_{a,1}} w_a(z_a) \tag{20}$$

also does *not* exist, and therefore $w_a(z_a)$ *cannot* be well defined at the corresponding point $z_{a,1}$. Therefore, if the singularity of $w_b(z_b)$ at $z_{b,1}$ is a hard one, then the singularity of $w_a(z_a)$ at $z_{a,1}$ must also be a hard singularity. This establishes Lemma 3.

Lemma 4: *If the singularity of $w_b(z_b)$ at $z_{b,1}$ is an integrable one, then the singularity of $w_a(z_a)$ at the corresponding singular point $z_{a,1}$ under the conformal transformation is also an integrable singularity.*

If the singularity of $w_b(z_b)$ at $z_{b,1}$ is an integrable one, then the integral of $w_b(z_b)$ along any open integration contour D_b going from a point $z_{b,0}$ internal to C_b to the singular point $z_{b,1}$,

$$\int_{D_b} dz_b\, w_b(z_b), \tag{21}$$

exists and is a finite complex number. We consider now the corresponding integral of $w_a(z_a)$ along an arbitrary open integration contour D_a going from an internal point $z_{a,0}$ within the open unit disk to the singular point $z_{a,1}$ on the unit circle, and make a transformation of the integration variable from z_a to z_b,

$$\begin{aligned} \int_{D_a} dz_a\, w_a(z_a) &= \int_{D_b} dz_b \left(\frac{dz_a}{dz_b}\right) w_b(z_b) \\ &= \int_{D_b} dz_b \left[\frac{d\gamma^{(-1)}(z_b)}{dz_b}\right] w_b(z_b), \end{aligned} \tag{22}$$

where we used the relations shown in Equations (14) and (16), where D_b is an open integration contour going from an internal point $z_{b,0}$ to the singular point $z_{b,1}$, and where D_b, $z_{b,0}$ and $z_{b,1}$ correspond respectively to D_a, $z_{a,0}$ and $z_{a,1}$, through the conformal transformation. Since the conformal transformation is an analytic function, the derivative within brackets is also an analytic function within and on D_b, and therefore is a limited function there. Since $w_b(z_b)$ by hypothesis is an integrable function around the singular point $z_{b,1}$, and since the product of a limited function and an integrable function is also an integrable function, the integrand of this last integral is an integrable function, and therefore this last integral exists and is a finite complex number. We thus conclude that

$$\int_{D_a} dz_a\, w_a(z_a) \tag{23}$$

exists and is a finite complex number. Therefore, $w_a(z_a)$ is an integrable function around the singular point $z_{a,1}$, and therefore that singularity is also an integrable one. This establishes Lemma 4.

We must now consider the question of what is the set of curves C_b for which the structure described above can be set up. Given the curve C_b, the only additional objects we need in order to do this is the conformal mapping $\gamma(z_a)$ and its inverse $\gamma^{(-1)}(z_b)$, between that curve and the unit circle C_a, as well as between the respective interiors. The existence of these transformation functions can be ensured as a consequence of the Riemann mapping theorem, and of the associated results relating to conformal mappings between regions of the complex plane [13]. According to that theorem, a conformal transformation such as the one we just described exists between any bounded simply connected open set of the plane and the open unit disk, and can be extended to the respective boundaries so long as the curve C_b is differentiable. There are therefore no additional limitations on the differentiable simple closed curves C_b that may be considered here.

5 Generalization to Arbitrary Analytic Functions

The basic idea of the proof of the generalization of the extended Cauchy-Goursat theorem will be to embed an arbitrarily given integration contour on the z_b plane, which must be a simple closed curve but may or may not be a differentiable curve, into a region bounded by a *differentiable* simple closed curve, which is then mapped to the unit circle on the z_a plane by a conformal transformation. The embedding will be such that all the isolated integrable singularities on the original integration contour are mapped onto the unit circle. This will then allow us to use the extended Cauchy-Goursat theorem for inner analytic functions within the unit disk, which was established in Section 3, to prove the generalized version of that theorem. Therefore, in this section we will prove the following theorem.

Theorem 2: *Given an analytic function $w(z)$, and a closed integration contour C within which $w(z)$ is analytic, and on which $w(z)$ is analytic except for a finite number of isolated integrable singularities, it follows that the integral of $w(z)$ over the contour C is zero,*

$$\oint_C dz\, w(z) = 0. \tag{24}$$

This is the most complete generalization of the extended Cauchy-Goursat theorem that we will consider here, the only relevant limitation being that the number of isolated integrable singularities be finite.

Proof 2.1:

Let $w_b(z_b)$ be an analytic function within a closed simple curve D_b on the z_b plane, and let the function $w_b(z_b)$ also be analytic almost everywhere *on* D_b, with the exception of a finite number of isolated integrable singular points. It follows that, due to Lemmas 1–4, the corresponding function $w_a(z_a)$ on the z_a plane will have the same number of singularities

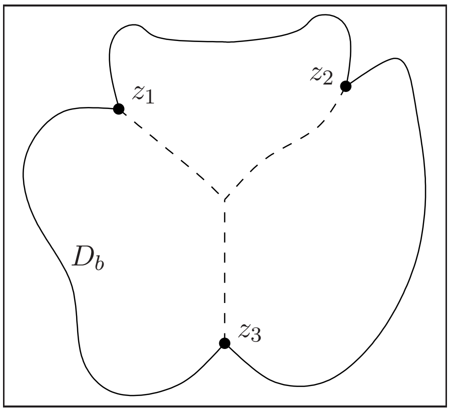

Figure 5: An integration contour D_b with three isolated singularities located at the points z_1, z_2 and z_3, where that contour is non-differentiable and concave, showing how to decompose it into three contours on which the singularities are located at points where the new contours are non-differentiable but convex, by the cuts shown (dashed lines).

on it, which will also be isolated integrable singular points. The curve D_b may not be differentiable at some points, including at some of the singular points. We will consider the integral of $w_b(z_b)$ over the integration contour D_b on the z_b plane, which will then, of course, correspond to the integral of $w_a(z_a)$ over a corresponding integration contour D_a on the z_a plane, under the conformal transformation.

The proof that follows will depend on the integration contour D_b being either differentiable or non-differentiable and *convex* at all the singular points found on it. However, this is not a true limitation, because an integration contour that has one of more singular points where it is non-differentiable and concave can always be decomposed into two or more integration contours where those same singular points are convex, as is shown in Figure 5 for an example with three such points. As one can see, all the three integration contours into which the original one is decomposed by the cuts shown (dashed lines) are convex at the singular points. When the three are put together to form once again the original contour, the integrals over those cuts, which due to their orientation are traversed once in each direction, cancel out. Therefore, if the theorem is proven for all contours which are convex at the singular points, it follows that it in fact holds for all contours, regardless of whether they are convex or concave at their singular points. We may therefore limit the proof to contours which are convex at all their singular points.

Given an integration contour D_b within which $w_b(z_b)$ is analytic, and on which $w_b(z_b)$ is analytic except for a finite number of isolated integrable singularities, at all of which the contour is either differentiable or non-differentiable and convex, we consider now the construction on the z_b plane of a new closed differentiable simple curve C_b that contains D_b. At any points on D_b where $w_b(z_b)$ is analytic there is a neighborhood of that point within which there are no singularities of $w_b(z_b)$ which are not located directly on D_b, whose union forms a strip around D_b. In this case we make C_b go through these neighborhoods in a differentiable fashion, outside the interior of D_b, so that we ensure that no singularities of $w_b(z_b)$ get included on C_b or in its interior, other than those on D_b. This can be done even at non-singular points where the integration contour D_b is *not* differentiable, in which case we make C_b just go around the point of non-differentiability of D_b, in a differentiable fashion, as can be seen illustrated in Figure 6.

At points of D_b where $w_b(z_b)$ has an isolated integrable singularity, since the singularity is isolated, there is also a neighborhood of the point within which there are no *other*

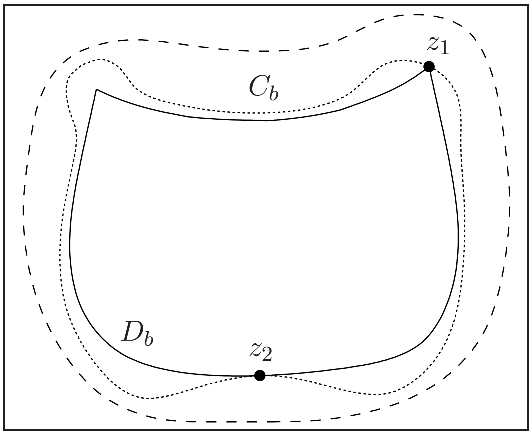

Figure 6: The construction a differentiable closed curve C_b containing the integration contour D_b, to be conformally mapped to the unit circle C_a, showing also the singular points z_1 and z_2, as well as a point of non-differentiability of the contour at which $w(z)$ is analytic.

singularities of $w_b(z_b)$, which is part of the aforementioned strip around D_b. In this case we make C_b go through this neighborhood, still keeping to the outer side of D_b, in such a way that the curve *runs over* the singular point in a differentiable way, which is possible because D_b is convex at that singular point, as is illustrated by the point z_1 in Figure 6. The singular point is one which the curve C_b will therefore share with D_b, as is also illustrated by the point z_1 in Figure 6. At singular points where D_b is differentiable we simply make C_b tangent to D_b at that point, as is illustrated by the point z_2 in Figure 6.

The result of this process, taken all around D_b and including all the isolated integrable singularities found on it, is a differentiable simple curve C_b which contains D_b and the singularities on it, but that contains no other singularities of $w_b(z_b)$, and which shares with D_b all the points where the relevant isolated integrable singularities of $w_b(z_b)$ are located. Since by construction C_b is a differentiable closed simple curve, by the Riemann mapping theorem there exists a conformal transformation $\gamma(z_a)$ that maps it from the unit circle. Therefore, the inverse conformal transformation $\gamma^{(-1)}(z_b)$ will map all the isolated integrable singular points on D_b to the unit circle C_a.

Since the interior of C_b in mapped by the inverse transformation $\gamma^{(-1)}(z_b)$ onto the open unit disk, it follows that the integration contour D_b, which is contained in C_b, is mapped by $\gamma^{(-1)}(z_b)$ onto a closed simple integration contour D_a contained in the unit disk in the z_a plane, which will not be differentiable if D_b is not, but which is contained within the closed unit disk, and that touches the unit circle only at each one of the isolated integrable singular points of $w_a(z_a)$ on D_a that correspond to the singularities of $w_b(z_b)$ on D_b.

Observe that, since the curve C_b does not contain any singularities of $w_b(z_b)$ in its strict interior, the interior of the curve C_a, which is the open unit disk, does not contain any singularities of $w_a(z_a)$. Therefore, according to the definition given in [1], $w_a(z_a)$ is an inner analytic function. If we now consider the integral of $w_b(z_b)$ over D_b, it is a very simple thing to change the integration variable from z_b to z_a,

$$\begin{aligned}
\oint_{D_b} dz_b\, w_b(z_b) &= \oint_{D_a} dz_a \left(\frac{dz_b}{dz_a}\right) w_a(z_a) \\
&= \oint_{D_a} dz_a \left[\frac{d\gamma(z_a)}{dz_a}\right] w_a(z_a),
\end{aligned} \tag{25}$$

where we used the relations shown in Equations (14) and (16). Because $\gamma(z_a)$ is an analytic function on the whole closed unit disk, the derivative in brackets is also an analytic function on the whole closed unit disk, and in addition to this the function $w_a(z_a)$ is analytic within the integration contour D_a and also on D_a except for a finite set of isolated singularities located on the unit circle. By the results of Lemmas 1–4, these isolated singularity are all integrable ones. Therefore, since the product of two analytic functions is also an analytic function, the integrand is analytic within the integration contour D_a, and also on it except for a finite set of isolated integrable singularities on the unit circle, and hence is an inner analytic function. Therefore, by Theorem 1, that is, the extended Cauchy-Goursat theorem for inner analytic functions on the unit disk, the integral is zero, and hence it follows that

$$\oint_{D_b} dz_b \, w_b(z_b) = 0. \tag{26}$$

In other words, due to the fact that the integral of $w_a(z_a)$ on D_a is zero, which is guaranteed by the extended Cauchy-Goursat theorem for inner analytic functions, we may conclude that the integral of $w_b(z_b)$ on D_b is also zero. This implies that the extended Cauchy-Goursat theorem is valid for $w_b(z_b)$, that is, for arbitrary complex analytic functions anywhere on the complex plane. This completes the proof of Theorem 2.

Note that, once we have Theorem 2 established, it is also valid for all inner analytic functions, and therefore automatically includes the contents of Theorem 1, which we may therefore regard as just an intermediate step in the proof.

6 Conclusions and Outlook

We have shown that an extended version of the Cauchy-Goursat theorem holds for all complex analytic functions, anywhere on the complex plane. The extension of the theorem establishes that the integral of any such function is zero, on any closed integration contour within which it is analytic, even it the function has a finite set of isolated singularities on the contour itself, so long as these are all integrable singularities. It is interesting to note that this integrability requirement on the singularities is the minimum necessary requirement for the integral over the contour to make any sense at all, and it is very curious that it turns out to be sufficient for the extension of the Cauchy-Goursat theorem.

The generalization of the result to infinite integration contours with a countable infinity of isolated integrable singularities on them is quite immediate, by a straightforward process of finite induction, using contour manipulation, the extended Cauchy-Goursat theorem and the Cauchy-Goursat theorem in its usual form. Therefore this extended Cauchy-Goursat theorem can be used in essentially all circumstances in which the original Cauchy-Goursat theorem applies, thus giving rise to extensions of many previously known results.

Note that the establishment of this extended Cauchy-Goursat theorem gives rise immediately to a corresponding extended set of Cauchy integral formulas, when the interior of the integration contours contain isolated poles. In fact, these extended Cauchy integral formulas are at the root of the representation of integrable real functions by inner analytic functions within the unit disk, which was established in [1]. In that paper the validity of these extended Cauchy integral formulas is reflected by the fact that integrals which give the coefficients of the Taylor series of the inner analytic functions are not only independent of the radius $0 < \rho < 1$ of the circle over which they are calculated, but are also continuous from within when one approaches the unit circle from within the open unit disk, that is, in the $\rho \to 1_{(-)}$ limit.

Since the Cauchy-Goursat theorem is such an important and fundamental one, it is to be expected that this extension will have further interesting consequences, possibly in many fields of mathematics and also in many applications in physics.

Acknowledgments

The author would like to thank his friend and colleague Prof. Carlos Eugênio Imbassay Carneiro, to whom he is deeply indebted for all his interest and help, as well as his careful reading of the manuscript and helpful criticism regarding this work.

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