Complex Analysis of Real Functions

VI: On the Convergence of Fourier Series

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Abstract

We define a compact version of the Hilbert transform, which we then use to write explicit expressions for the partial sums and remainders of arbitrary Fourier series. The expression for the partial sums reproduces the known result in terms of Dirichlet integrals. The expression for the remainder is written in terms of a similar type of integral. Since the asymptotic limit of the remainder being zero is a necessary and sufficient condition for the convergence of the series, this same condition on the asymptotic behavior of the corresponding integrals constitutes such a necessary and sufficient condition.

1 Introduction

In a previous paper [1] we introduced a certain complex-analytic structure within the unit disk of the complex plane, and showed that it is possible to represent within that structure essentially *all* integrable real functions defined in a compact interval. In a subsequent paper [2] we showed that *all* the elements of the Fourier theory [3] of integrable real functions are contained within that complex-analytic structure. However, in that paper we did not discuss in any depth the question of the convergence of Fourier series.

The fact that it is possible to recover the real functions from their Fourier coefficients almost everywhere, even when the corresponding Fourier series are divergent, as we showed in [1], led to a powerful and very general summation rule for *all* Fourier series, which was presented in [2]. This summation rule allows one to add up a regularized version of the Fourier series, in a meaningful way, and therefore allows one to simply circumvent the fact that the original Fourier series may be divergent. However, the complex-analytic structure actually does allow for a direct discussion of the convergence problem.

In this paper we will present a more complete analysis of the convergence of Fourier series. In order to do this we will first introduce what we will name the *compact Hilbert transform*, which is a version of the Hilbert transform which is appropriate for functions defined on a compact interval. This will lead not only to the known explicit expression for the partial sums of the Fourier series in terms of Dirichlet integrals, but also to an explicit expression for the *remainder* of the Fourier series, in terms of a similar type of integral.

For ease of reference, we include here a one-page synopsis of the complex-analytic structure introduced in [1]. It consists of certain elements within complex analysis [4], as well as of their main properties.

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Synopsis: The Complex-Analytic Structure

An *inner analytic function* $w(z)$ is simply a complex function which is analytic within the open unit disk. An inner analytic function that has the additional property that $w(0) = 0$ is a *proper inner analytic function.* The *angular derivative* of an inner analytic function is defined by

$$w^{\cdot}(z) = \imath z \, \frac{dw(z)}{dz}. \tag{1}$$

By construction we have that $w^{\cdot}(0) = 0$, for all $w(z)$. The *angular primitive* of an inner analytic function is defined by

$$w^{-1\cdot}(z) = -\imath \int_0^z dz' \, \frac{w(z') - w(0)}{z'}. \tag{2}$$

By construction we have that $w^{-1\cdot}(0) = 0$, for all $w(z)$. In terms of a system of polar coordinates (ρ, θ) on the complex plane, these two analytic operations are equivalent to differentiation and integration with respect to θ, taken at constant ρ. These two operations stay within the space of inner analytic functions, they also stay within the space of proper inner analytic functions, and they are the inverses of one another. Using these operations, and starting from any proper inner analytic function $w^{0\cdot}(z)$, one constructs an infinite *integral-differential chain* of proper inner analytic functions,

$$\left\{ \ldots, w^{-3\cdot}(z), w^{-2\cdot}(z), w^{-1\cdot}(z), w^{0\cdot}(z), w^{1\cdot}(z), w^{2\cdot}(z), w^{3\cdot}(z), \ldots \right\}. \tag{3}$$

Two different such integral-differential chains cannot ever intersect each other. There is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, namely the null chain, in which all members are the null function $w(z) \equiv 0$.

A general scheme for the classification of all possible singularities of inner analytic functions is established. A singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity.* Angular integration takes soft singularities to other soft singularities, and angular differentiation takes hard singularities to other hard singularities.

Gradations of softness and hardness are then established. A hard singularity that becomes a soft one by means of a single angular integration is a *borderline hard* singularity, with degree of hardness zero. The *degree of softness* of a soft singularity is the number of angular differentiations that result in a borderline hard singularity, and the *degree of hardness* of a hard singularity is the number of angular integrations that result in a borderline hard singularity. Singularities which are either soft or borderline hard are integrable ones. Hard singularities which are not borderline hard are non-integrable ones.

Given an integrable real function $f(\theta)$ on the unit circle, one can construct from it a unique corresponding inner analytic function $w(z)$. The real function $f(\theta)$ is recovered by means of the $\rho \to 1_{(-)}$ limit of the real part of this inner analytic function. Singularities of real functions can be classified in a way which is analogous to the corresponding complex classification. Integrable real functions are typically associated with inner analytic functions that have singularities which are either soft or at most borderline hard. A more detailed review of real functions will be given in Section 2. This ends our synopsis.

Some of the material contained in this paper can be seen as a development, reorganization and extension of some of the material found, sometimes still in rather rudimentary form, in the papers [5–9].

2 Review of Real Functions

When we discuss real functions in this paper, some properties will be globally assumed for these functions, just as was done in [1, 2, 10]. These are rather weak conditions to be imposed on these functions, that will be in force throughout this paper. It is to be understood, without any need for further comment, that these conditions are valid whenever real functions appear in the arguments. These weak conditions certainly hold for any integrable real functions that are obtained as restrictions of corresponding inner analytic functions to the unit circle.

The most basic condition is that the real functions must be measurable in the sense of Lebesgue, with the usual Lebesgue measure [11, 12]. The second global condition we will impose is that the functions have no removable singularities. The third and last global condition is that the number of hard singularities on the unit circle be finite, and hence that they be all isolated from one another. There will be no limitation on the number of soft singularities.

In addition to this we will assume, for the purposes of this particular paper, that all real functions are integrable on the unit circle and, just for the sake of clarity and simplicity, unless explicitly stated otherwise we will also assume that all real functions are zero-average real functions, meaning that their integrals over the unit circle are zero. Since this simply implies that the Fourier coefficients α_0 of the real functions are zero, without affecting any of the other coefficients in any way, this clearly has no impact on any arguments about the convergence of the Fourier series.

For the purposes of this paper it is important to review here, in some detail, the construction that results in the correspondence between integrable real functions on the unit circle and inner analytic functions on the open unit disk. In [1] we showed that, given any integrable real function $f(\theta)$, one can construct a corresponding inner analytic function $w(z)$, from the real part of which $f(\theta)$ can be recovered almost everywhere on the unit circle, through the use of the $\rho \to 1_{(-)}$ limit, where (ρ, θ) are polar coordinates on the complex plane. In that construction we started by calculating the Fourier coefficients [3] α_k and β_k of the real function, which is always possible given that the function is integrable, using the usual integrals defining these coefficients,

$$\begin{aligned} \alpha_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, f(\theta), \\ \alpha_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f(\theta), \\ \beta_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f(\theta), \end{aligned} \tag{4}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. We then defined a set of complex Taylor coefficients c_k given by

$$\begin{aligned} c_0 &= \frac{1}{2} \alpha_0, \\ c_k &= \alpha_k - \imath \beta_k, \end{aligned} \tag{5}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. Next we defined a complex variable z associated to θ, using the positive real variable ρ, by $z = \rho \exp(\imath\theta)$. Using all these elements we then constructed the complex power series

$$S(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{6}$$

which we showed in [1] to be convergent to an inner analytic function $w(z)$ within the open unit disk. That inner analytic function may be written as

$$w(z) = u(\rho, \theta) + \imath v(\rho, \theta). \tag{7}$$

The complex power series in Equation (6) is therefore the Taylor series of $w(z)$. We also proved in [1] that one recovers the real function $f(\theta)$ almost everywhere on the unit circle from the real part $u(\rho, \theta)$ of $w(z)$, by means of the $\rho \rightarrow 1_{(-)}$ limit. The $\rho \rightarrow 1_{(-)}$ limit of the imaginary part $v(\rho, \theta)$ also exists almost everywhere and gives rise to a real function $g(\theta)$ which corresponds to $f(\theta)$. The pair of real functions obtained from the real and imaginary parts of one and the same inner analytic function are said to be mutually Fourier-conjugate real functions.

In a subsequent paper [2] we showed that *all* the elements of the Fourier theory [3] of integrable real functions are contained within the complex-analytic structure, including the Fourier basis of functions, the Fourier series, the scalar product for integrable real functions, the relations of orthogonality and norm of the basis elements, and the completeness of the Fourier basis, including its so-called completeness relation. As was also shown in [2] the real function $g(\theta) = v(1, \theta)$ which is the Fourier-conjugate function to $f(\theta) = u(1, \theta)$ has the same Fourier coefficients, but with the meanings of α_k and β_k interchanged in such a way that we have

$$\begin{aligned} \alpha_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) g(\theta), \\ \beta_k &= -\frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, \cos(k\theta) g(\theta), \end{aligned} \tag{8}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. Note that there is no constant term in the Fourier series of $g(\theta)$, which means that we have

$$\int_{-\pi}^{\pi} d\theta\, g(\theta) = 0. \tag{9}$$

In other words, the Fourier-conjugate function $g(\theta)$ is always a zero-average real function. Note also that this fact, as well as the relations in Equation (8) imply, in particular, that $g(\theta)$ is also an integrable real function. We may therefore conclude that, if $f(\theta)$ is an integrable real function, then so is its Fourier-conjugate function $g(\theta)$.

3 The Compact Hilbert Transform

Let $f(\theta)$ be an integrable real function on $[-\pi, \pi]$, with Fourier coefficients as given in Equation (4). As was shown in [2] the real function $g(\theta) = v(1, \theta)$ which is the Fourier-conjugate function to $f(\theta) = u(1, \theta)$ has the same Fourier coefficients, but with the meanings of α_k and β_k interchanged, as shown in Equation (8). The relations in Equations (4) and (8) can be understood as the following collection of integral identities satisfied by all pairs of Fourier-conjugate integrable real functions,

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f(\theta) &= \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) g(\theta), \\ \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f(\theta) &= -\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) g(\theta), \end{aligned} \tag{10}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. It is well known that this replacement of $\cos(k\theta)$ with $\sin(k\theta)$ and of $\sin(k\theta)$ with $-\cos(k\theta)$ can be effected by the use of the Hilbert transform. However, that transform was originally introduced by Hilbert for real functions defined on the whole real line, rather that on the unit circle as is our case here. Therefore, the first thing that we will do here is to define a compact version of the Hilbert transform that applies to real functions defined on the unit circle.

Since the Fourier coefficient α_0 of $f(\theta)$ has no effect on the definition of the Fourier-conjugate function $g(\theta)$, and in order for this pair of real functions to be related in a unique way, we will assume that $f(\theta)$ is also a zero-average real function,

$$\int_{-\pi}^{\pi} d\theta\, f(\theta) = 0, \tag{11}$$

thus implying for its $k = 0$ Fourier coefficient that $\alpha_0 = 0$. This does not affect, of course, any subsequent arguments about the convergence of the Fourier series. According to the construction presented in [1] and reviewed in Section 2, from the other Fourier coefficients α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, we may construct the complex coefficients $c_k = \alpha_k - \imath\beta_k$, for $k \in \{1, 2, 3, \ldots, \infty\}$, where we now have $c_0 = 0$, and from these we may construct the corresponding inner analytic function $w(z)$ shown in Equation (7), which is now, in fact, a *proper* inner analytic function, since $c_0 = 0$ implies that $w(0) = 0$.

Since $u(\rho, \theta)$ and $v(\rho, \theta)$ are harmonic conjugate functions to each other, it is now clear that there is a one-to-one correspondence between $u(\rho, \theta)$ and $v(\rho, \theta)$, and in particular between $u(1, \theta)$ and $v(1, \theta)$. Therefore, there is a one-to-one correspondence between $f(\theta)$ and $g(\theta)$, in this case valid almost everywhere on the unit circle, since we have shown in [1] that $f(\theta) = u(1, \theta)$ and that $g(\theta) = v(1, \theta)$, both almost everywhere over the unit circle. Therefore, a transformation must exist that produces $g(\theta)$ from $f(\theta)$ almost everywhere over the unit circle, as well as an inverse transformation that recovers $f(\theta)$ from $g(\theta)$ almost everywhere over the unit circle. In this section we will show that the following definition accomplishes this.

Definition 1: *Compact Hilbert Transform*

If $f(\theta)$ is an arbitrarily given zero-average integrable real function defined on the unit circle, then its *compact Hilbert transform* $g(\theta)$ is the real function defined by

$$\begin{aligned} g(\theta) &= \mathcal{H}_{\rm c}[f(\theta)] \\ &= -\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\, \frac{\cos\!\left[(\theta_1-\theta)/2\right]}{\sin\!\left[(\theta_1-\theta)/2\right]}\, f(\theta_1), \end{aligned} \tag{12}$$

where PV stands for the Cauchy principal value, and where $\mathcal{H}_{\rm c}[f(\theta)]$ is the notation we will use for the compact Hilbert transform applied to the real function $f(\theta)$.

We will now prove the following theorem.

Theorem 1: *The zero-average integrable real functions $f(\theta)$ and $g(\theta)$, which are such that $f(\theta) = u(1, \theta)$ and $g(\theta) = v(1, \theta)$ almost everywhere on the unit circle, are related to each other by this transform, that is, we have that $g(\theta) = \mathcal{H}_{\rm c}[f(\theta)]$ almost everywhere on the unit circle, and that $f(\theta) = \mathcal{H}_{\rm c}^{-1}[g(\theta)]$ almost everywhere on the unit circle, where the inverse transform is simply given by $\mathcal{H}_{\rm c}^{-1}[g(\theta)] = -\mathcal{H}_{\rm c}[g(\theta)]$.*

Proof 1.1:

In order to derive these facts from our complex-analytic structure, we start from the Cauchy integral formula for the inner analytic function $w(z)$,

$$w(z) = \frac{1}{2\pi\imath}\oint_C dz_1\,\frac{w(z_1)}{z_1 - z}, \tag{13}$$

where C can be taken as a circle centered at the origin, with radius $\rho_1 < 1$, and where we write z and z_1 in polar coordinates as $z = \rho\exp(\imath\theta)$ and $z_1 = \rho_1\exp(\imath\theta_1)$. The integral formula in Equation (13) is valid for $\rho < \rho_1$, and in fact, by the Cauchy-Goursat theorem, the integral is zero if $\rho > \rho_1$, since both z and z_1 are within the open unit disk, a region where $w(z)$ is analytic. We must now determine what happens if $\rho = \rho_1$, that is, if z is on the circle C of radius ρ_1. Note that in this case we may slightly deform the integration contour C in order to have it pass on one side or the other of the simple pole of the integrand at $z_1 = z$. If we use a deformed contour $C_\ominus$ that *excludes* the pole from its interior, then we have, instead of Equation (13),

$$0 = \frac{1}{2\pi\imath}\oint_{C_\ominus} dz_1\,\frac{w(z_1)}{z_1 - z}, \tag{14}$$

due to the Cauchy-Goursat theorem, while if we use a deformed contour $C_\oplus$ that *includes* the pole in its interior, then we have, just as in Equation (13),

$$w(z) = \frac{1}{2\pi\imath}\oint_{C_\oplus} dz_1\,\frac{w(z_1)}{z_1 - z}. \tag{15}$$

Since by the Sokhotskii-Plemelj theorem [13] the Cauchy principal value of the integral over the circle C is the arithmetic average of these two integrals, in the limit where the deformations vanish, a limit which does not really have to be considered in detail, so long as the deformations do not cross any other singularities,

$$\mathrm{PV}\!\oint_C dz_1\,\frac{w(z_1)}{z_1 - z} = \frac{1}{2}\oint_{C_\ominus} dz_1\,\frac{w(z_1)}{z_1 - z} + \frac{1}{2}\oint_{C_\oplus} dz_1\,\frac{w(z_1)}{z_1 - z}, \tag{16}$$

adding Equations (14) and (15) we may conclude that

$$w(z) = \frac{1}{\pi\imath}\,\mathrm{PV}\!\oint_C dz_1\,\frac{w(z_1)}{z_1 - z}, \tag{17}$$

where we now have $\rho = \rho_1$, that is, both z_1 and z are on the circle C of radius ρ_1 within the open unit disk. This formula can be understood as a special version of the Cauchy integral formula, and will be used repeatedly in what follows. We may now write all quantities in this equation in terms of the polar coordinates ρ_1, θ_1 and θ,

$$\begin{aligned} w(\rho_1,\theta) &= \frac{1}{\pi\imath}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\imath\rho_1\,\mathrm{e}^{\imath\theta_1}\,\frac{w(\rho_1,\theta_1)}{\rho_1\,\mathrm{e}^{\imath\theta_1} - \rho_1\,\mathrm{e}^{\imath\theta}} \\ &= \frac{1}{\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{u(\rho_1,\theta_1) + \imath v(\rho_1,\theta_1)}{1 - \mathrm{e}^{-\imath\Delta\theta}}, \end{aligned} \tag{18}$$

where $\Delta\theta = \theta_1 - \theta$. Note that, since by construction the real and imaginary parts $u(\rho_1,\theta)$ and $v(\rho_1,\theta)$ of $w(\rho_1,\theta)$ for $\rho_1 = 1$ are integrable real functions on the unit circle, and since there are no other dependencies on ρ_1 in this expression, we may now take the $\rho_1 \to 1_{(-)}$ limit of this equation, in which the principal value acquires its usual real meaning over the unit circle, that is, the meaning that the asymptotic limits of the integral on either side of a non-integrable singularity must be taken in the symmetric way. In that limit we have

$$u(1,\theta)+\imath v(1,\theta)=\frac{1}{\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{u(1,\theta_1)+\imath v(1,\theta_1)}{1-\mathrm{e}^{-\imath\Delta\theta}}. \tag{19}$$

In order to identify separately the real and imaginary parts of this equation, we must now rationalize the integrand of the integral shown. We will use the fact that

$$\begin{aligned}
\frac{1}{1-\mathrm{e}^{-\imath\Delta\theta}} &= \frac{1-\mathrm{e}^{\imath\Delta\theta}}{\left(1-\mathrm{e}^{-\imath\Delta\theta}\right)\left(1-\mathrm{e}^{\imath\Delta\theta}\right)}\\
&= \frac{\left[1-\cos(\Delta\theta)\right]-\imath\sin(\Delta\theta)}{2-2\cos(\Delta\theta)}\\
&= \frac{1}{2}\left[1-\imath\,\frac{\sin(\Delta\theta)}{1-\cos(\Delta\theta)}\right].
\end{aligned} \tag{20}$$

Using the half-angle trigonometric identities we may also write this result as

$$\frac{1}{1-\mathrm{e}^{-\imath\Delta\theta}} = \frac{1}{2}\left[1-\imath\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\right] \tag{21}$$

$$= -\frac{\imath}{2}\,\frac{\mathrm{e}^{\imath\Delta\theta/2}}{\sin(\Delta\theta/2)}. \tag{22}$$

Using the result shown in Equation (21) back in Equation (19) we obtain

$$\begin{aligned}
u(1,\theta)+\imath v(1,\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\left[u(1,\theta_1)+\imath v(1,\theta_1)\right]\left[1-\imath\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\right]\\
&= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\left[u(1,\theta_1)+\imath v(1,\theta_1)\right]+\\
&\quad +\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\left[v(1,\theta_1)-\imath u(1,\theta_1)\right].
\end{aligned} \tag{23}$$

Since both $u(1,\theta_1)$ and $v(1,\theta_1)$ are zero-average real functions on the unit circle, the first two integrals in the last form of the equation above are zero, so that separating the real and imaginary parts within this expression we are left with

$$\begin{aligned}
u(1,\theta)+\imath v(1,\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,v(1,\theta_1)+\\
&\quad -\frac{\imath}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,u(1,\theta_1),
\end{aligned} \tag{24}$$

where $\Delta\theta=\theta_1-\theta$. Separating the real and imaginary parts of this equation we may now write that

$$\begin{aligned}
u(1,\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,v(1,\theta_1),\\
v(1,\theta) &= -\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,u(1,\theta_1),
\end{aligned} \tag{25}$$

where $\Delta\theta=\theta_1-\theta$. Recalling that $f(\theta)=u(1,\theta)$ and $g(\theta)=v(1,\theta)$, almost everywhere on the unit circle, we have

$$
\begin{aligned}
f(\theta) &= \mathcal{H}_{\rm c}^{-1}[g(\theta)] \\
&= \frac{1}{2\pi}\,{\rm PV}\!\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,g(\theta_1), \\
g(\theta) &= \mathcal{H}_{\rm c}[f(\theta)] \\
&= -\frac{1}{2\pi}\,{\rm PV}\!\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,f(\theta_1),
\end{aligned}
\tag{26}
$$

two equations which are thus valid almost everywhere as well. These are the transformations relating the pair of Fourier-conjugate functions $f(\theta)$ and $g(\theta)$. The second expression defines the compact Hilbert transformation of $f(\theta)$ into $g(\theta)$, and the first one defines the inverse transformation, which recovers $f(\theta)$ from $g(\theta)$. Note that in this notation the transform is defined with an explicit minus sign, and that its inverse is simply minus the transform itself, $\mathcal{H}_{\rm c}^{-1}[g(\theta)] = -\mathcal{H}_{\rm c}[g(\theta)]$. This completes the proof of Theorem 1.

It is interesting to observe that this transform can be interpreted as a linear integral operator acting on the space of zero-average integrable real functions defined on the unit circle. The integration kernel of the integral operator depends only on the difference $\theta - \theta_1$, and is given by

$$
K_{\mathcal{H}_{\rm c}}(\theta-\theta_1) = -\frac{1}{2\pi}\,\frac{\cos\left[(\theta_1-\theta)/2\right]}{\sin\left[(\theta_1-\theta)/2\right]},
\tag{27}
$$

so that the action of the operator on an arbitrarily given zero-average real integrable function $f(\theta)$ on the unit circle can be written as

$$
\begin{aligned}
g(\theta) &= \mathcal{H}_{\rm c}[f(\theta)] \\
&= {\rm PV}\!\!\int_{-\pi}^{\pi} d\theta_1\,K_{\mathcal{H}_{\rm c}}(\theta-\theta_1)f(\theta_1).
\end{aligned}
\tag{28}
$$

The operator is linear, invertible, and the composition of the operator with itself results in the operation of multiplication by -1. Note that, since by hypothesis $f(\theta)$ is integrable on the unit circle, the Cauchy principal value refers only to the explicit non-integrable singularity of the integration kernel at the position $\theta_1 = \theta$.

4 Action on the Fourier Basis

We will now determine the action of the compact Hilbert transform on the elements of the Fourier basis of functions. The case of the constant function, which constitutes the $k = 0$ element of the basis, that is the single member of the basis which is not a zero-average function, must be examined in separate. We will now prove the following simple theorem.

Theorem 2: *Given any constant real function $f(\theta) = R$, for any real constant R, its compact Hilbert transform is zero, that is, $\mathcal{H}_{\rm c}[R] = 0$.*

Proof 2.1:

We start from the expression in Equation (17) for the very simple case $w(z) = 1$,

$$
1 = \frac{1}{\pi \imath}\,{\rm PV}\!\oint_C dz_1\,\frac{1}{z_1 - z},
\tag{29}
$$

where both z_1 and z are on the circle C of radius ρ_1. We may now write all quantities in this equation in terms of ρ_1, θ_1 and θ,

$$\begin{aligned} 1 &= \frac{1}{\pi\imath}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\, \imath\rho_1\,\mathrm{e}^{\imath\theta_1}\,\frac{1}{\rho_1\,\mathrm{e}^{\imath\theta_1}-\rho_1\,\mathrm{e}^{\imath\theta}} \\ &= \frac{1}{\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{1}{1-\,\mathrm{e}^{-\imath\Delta\theta}}, \end{aligned} \tag{30}$$

where $\Delta\theta=\theta_1-\theta$. Note that, since there are no remaining dependencies on ρ_1, we may now take the $\rho_1\to 1_{(-)}$ limit of this expression, in which the principal value acquires its usual real meaning on the unit circle. Just as in the previous section, in order to identify separately the real and imaginary parts of this equation, we must now rationalize the integrand. Using the result in Equation (21) we obtain

$$\begin{aligned} 1 &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\left[1-\imath\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\right] \\ &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\;-\;\frac{\imath}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)} \\ &= 1-\frac{\imath}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}. \end{aligned} \tag{31}$$

It follows therefore that we have

$$-\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}=0, \tag{32}$$

which is the statement that $\mathcal{H}_{\rm c}[1]=0$. Note that, since $\Delta\theta=\theta_1-\theta$, which in the context of this integral implies that $d\theta_1=d(\Delta\theta)$, by means of a trivial transformation of variables this integral can also be shown to be zero by simple parity arguments. Given the linearity of the compact Hilbert transform, it is equally true that, for any real constant R, we have that $\mathcal{H}_{\rm c}[R]=0$, so that all constant functions are mapped to the null function. This completes the proof of Theorem 2.

Let us now consider all the remaining elements of the Fourier basis of functions. We will prove the following theorem.

Theorem 3: *Given the elements of the Fourier basis of functions,* $\cos(k\theta)$ *and* $\sin(k\theta)$*, for* $k\in\{1,2,3,\ldots,\infty\}$*, the following relations between them hold:*

$$\begin{aligned} \cos(k\theta) &= -\mathcal{H}_{\rm c}\big[\sin(k\theta)\big]\,, \\ \sin(k\theta) &= \mathcal{H}_{\rm c}\big[\cos(k\theta)\big]\,. \end{aligned} \tag{33}$$

Proof 3.1:

In order to prove this theorem we start from the expression in Equation (17) for the case $w(z)=z^k$, where $k\in\{1,2,3,\ldots,\infty\}$, that is, for a strictly positive power of z, which is therefore an inner analytic function. Note that these are all the elements of the complex Taylor basis of functions, with the exception of the constant function. We have therefore

$$z^k=\frac{1}{\pi\imath}\,\mathrm{PV}\!\oint_C dz_1\,\frac{z_1^k}{z_1-z}, \tag{34}$$

where both z_1 and z are on the circle C of radius ρ_1. We may now write all quantities in this equation in terms of ρ_1, θ_1 and θ,

$$\begin{aligned}
\rho_1^k\,\mathrm{e}^{\imath k\theta} &= \frac{1}{\pi\imath}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\imath\rho_1\,\mathrm{e}^{\imath\theta_1}\,\frac{\rho_1^k\,\mathrm{e}^{\imath k\theta_1}}{\rho_1\,\mathrm{e}^{\imath\theta_1}-\rho_1\,\mathrm{e}^{\imath\theta}} \quad\Rightarrow\\
\mathrm{e}^{\imath k\theta} &= \frac{1}{\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\mathrm{e}^{\imath k\theta_1}}{1-\,\mathrm{e}^{-\imath\Delta\theta}}, \qquad (35)
\end{aligned}$$

where $\Delta\theta=\theta_1-\theta$. Note that, since there are no remaining dependencies on ρ_1, we may now take the $\rho_1\to 1_{(-)}$ limit of this expression, in which the principal value acquires its usual real meaning on the unit circle. In the limit we have

$$\cos(k\theta)+\imath\sin(k\theta)=\frac{1}{\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(k\theta_1)+\imath\sin(k\theta_1)}{1-\,\mathrm{e}^{-\imath\Delta\theta}}. \qquad (36)$$

Just as in the previous cases, in order to identify separately the real and imaginary parts of this equation, we must now rationalize the integrand. Using the result in Equation (21) we obtain

$$\begin{aligned}
&\cos(k\theta)+\imath\sin(k\theta)\\
&\quad= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\big[\cos(k\theta_1)+\imath\sin(k\theta_1)\big]\left[1-\imath\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\right]\\
&\quad= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\big[\cos(k\theta_1)+\imath\sin(k\theta_1)\big]+\\
&\qquad+\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\big[\sin(k\theta_1)-\imath\cos(k\theta_1)\big]\,, \qquad (37)
\end{aligned}$$

where $\Delta\theta=\theta_1-\theta$ and $k\in\{1,2,3,\ldots,\infty\}$. The first two integrals in the last form of the equation above are zero for all $k>0$ because they are integrals of cosines and sines over integer multiples of their periods, so that we may now separate the real and imaginary parts of the remaining terms and thus get

$$\begin{aligned}
\cos(k\theta)+\imath\sin(k\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,\sin(k\theta_1)+\\
&\quad-\frac{\imath}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,\cos(k\theta_1), \qquad (38)
\end{aligned}$$

where $\Delta\theta=\theta_1-\theta$ and $k\in\{1,2,3,\ldots,\infty\}$. We therefore obtain the action of the compact Hilbert transform on the elements of the Fourier basis,

$$\begin{aligned}
\cos(k\theta) &= -\mathcal{H}_{\mathrm{c}}\big[\sin(k\theta)\big]\\
&= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,\sin(k\theta_1),\\
\sin(k\theta) &= \mathcal{H}_{\mathrm{c}}\big[\cos(k\theta)\big]\\
&= -\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\cos(\Delta\theta/2)}{\sin(\Delta\theta/2)}\,\cos(k\theta_1), \qquad (39)
\end{aligned}$$

where $\Delta\theta=\theta_1-\theta$ and $k\in\{1,2,3,\ldots,\infty\}$. The second equation above is the transform applied to the cosines and the first equation is the inverse transform applied to the sines. As one can see, the transform does indeed have the property of replacing cosines with sines and sines with minus cosines, as expected. This completes the proof of Theorem 3.

One can now see that the application of the compact Hilbert transform to the Fourier series of an arbitrarily given zero-average integrable real function $f(\theta)$ on the unit circle will produce the Fourier series of its Fourier-conjugate real function $g(\theta)$. Given the linearity of the transform, if we apply it to the Fourier series of $f(\theta)$ we get

$$\begin{aligned}
\mathcal{H}_{\rm c}\left\{\sum_{k=1}^{\infty}\left[\alpha_k\cos(k\theta)+\beta_k\sin(k\theta)\right]\right\} &= \sum_{k=1}^{\infty}\left\{\alpha_k\mathcal{H}_{\rm c}\left[\cos(k\theta)\right]+\beta_k\mathcal{H}_{\rm c}\left[\sin(k\theta)\right]\right\} \\
&= \sum_{k=1}^{\infty}\left[\alpha_k\sin(k\theta)-\beta_k\cos(k\theta)\right], \qquad (40)
\end{aligned}$$

where this last one is the Fourier series of the real function $g(\theta)$, which is the Fourier conjugate of $f(\theta)$. Hence, if $S(\rho,\theta)$ is the complex power series given in Equation (6), if $S^{F,f}(\theta)=\Re[S(1,\theta)]$ is the Fourier series of $f(\theta)$ and $S^{F,g}(\theta)=\Im[S(1,\theta)]$ is the Fourier series of $g(\theta)$, then we have that

$$S^{F,g}(\theta)=\mathcal{H}_{\rm c}\left[S^{F,f}(\theta)\right]. \qquad (41)$$

Note that the same is true for the corresponding partial sums, as well as for the corresponding remainders, so long as the latter exist at all. If $S_N(\rho,\theta)$ is the $N^{\rm th}$ partial sum and $R_N(\rho,\theta)$ is the $N^{\rm th}$ remainder of the complex power series given in Equation (6), and if $S_N^{F,f}(\theta)=\Re[S_N(1,\theta)]$ is the $N^{\rm th}$ partial sum of the real Fourier series of $f(\theta)$, if $S_N^{F,g}(\theta)=\Im[S_N(1,\theta)]$ is the $N^{\rm th}$ partial sum of the Fourier series of $g(\theta)$, if $R_N^{F,f}(\theta)=\Re[R_N(1,\theta)]$ is the $N^{\rm th}$ remainder of the Fourier series of $f(\theta)$ and if $R_N^{F,g}(\theta)=\Im[R_N(1,\theta)]$ is the $N^{\rm th}$ remainder of the Fourier series of $g(\theta)$, then we have

$$\begin{aligned}
S_N^{F,g}(\theta) &= \mathcal{H}_{\rm c}\left[S_N^{F,f}(\theta)\right], \\
R_N^{F,g}(\theta) &= \mathcal{H}_{\rm c}\left[R_N^{F,f}(\theta)\right]. \qquad (42)
\end{aligned}$$

Of course, in each one of these cases the inverse mapping holds as well, using the inverse transform to take us from the quantities related to $g(\theta)$ back to the corresponding quantities related to $f(\theta)$.

5 An Infinite Collection of Identities

In order to obtain a certain infinite collection of identities satisfied by all zero-average integrable real functions and their Fourier-conjugate real functions, which will be very important later, we start by examining the action of the compact Hilbert transform on the products of arbitrarily given integrable real functions and the elements of the Fourier basis. We will prove the following theorem.

Theorem 4: *Given an arbitrary zero-average integrable real function $f(\theta)$ on the unit circle, and the corresponding Fourier-conjugate real function $g(\theta)$, these two real functions satisfy almost everywhere the following infinite collection of identities:*

$$\begin{aligned}
f(\theta) &= \frac{1}{2\pi}\,{\rm PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\sin\left[(k+1/2)\Delta\theta\right]f(\theta_1)+\cos\left[(k+1/2)\Delta\theta\right]g(\theta_1)}{\sin(\Delta\theta/2)}, \\
g(\theta) &= \frac{1}{2\pi}\,{\rm PV}\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\sin\left[(k+1/2)\Delta\theta\right]g(\theta_1)-\cos\left[(k+1/2)\Delta\theta\right]f(\theta_1)}{\sin(\Delta\theta/2)}, \qquad (43)
\end{aligned}$$

where $\Delta\theta=\theta_1-\theta$ and $k\in\{1,2,3,\ldots,\infty\}$.

Proof 4.1:

In order to prove this theorem we start from the expression in Equation (17), exchanging $w(z)$ for the product $z^k w(z)$, which is also an inner analytic function so long as z^k is an arbitrary positive integer power, which it is since we assume that $k \in \{1, 2, 3, \ldots, \infty\}$. We therefore have

$$z^k w(z) = \frac{1}{\pi \imath} \mathrm{PV}\oint_C dz_1 \, \frac{z_1^k w(z_1)}{z_1 - z}, \tag{44}$$

where both z_1 and z are on the circle C of radius ρ_1 within the open unit disk. We may now write all quantities in this equation in terms of ρ_1, θ_1 and θ,

$$\begin{aligned} \rho_1^k \, \mathrm{e}^{\imath k\theta} w(\rho_1, \theta) &= \frac{1}{\pi \imath} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \imath \rho_1 \, \mathrm{e}^{\imath\theta_1} \, \frac{\rho_1^k \, \mathrm{e}^{\imath k\theta_1} w(\rho_1, \theta_1)}{\rho_1 \, \mathrm{e}^{\imath\theta_1} - \rho_1 \, \mathrm{e}^{\imath\theta}} \quad \Rightarrow \\ w(\rho_1, \theta) &= \frac{1}{\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \frac{\mathrm{e}^{\imath k\Delta\theta} w(\rho_1, \theta_1)}{1 - \mathrm{e}^{-\imath\Delta\theta}}, \end{aligned} \tag{45}$$

where $\Delta\theta = \theta_1 - \theta$ and $k \in \{1, 2, 3, \ldots, \infty\}$. Note that, since by construction the real and imaginary parts $u(\rho_1, \theta)$ and $v(\rho_1, \theta)$ of $w(\rho_1, \theta)$ for $\rho_1 = 1$ are integrable real functions on the unit circle, and since there are no other dependencies on ρ_1 in this equation, we may now take the $\rho_1 \to 1_{(-)}$ limit of this expression, in which the principal value acquires its usual real meaning on the unit circle, thus obtaining

$$u(1, \theta) + \imath v(1, \theta) = \frac{1}{\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \frac{\mathrm{e}^{\imath k\Delta\theta} \left[u(1, \theta_1) + \imath v(1, \theta_1) \right]}{1 - \mathrm{e}^{-\imath\Delta\theta}}, \tag{46}$$

where $\Delta\theta = \theta_1 - \theta$ and $k \in \{1, 2, 3, \ldots, \infty\}$. Once more, in order to identify separately the real and imaginary parts of this equation, we must now rationalize the integrand. Using this time the form shown in Equation (22) for the factor to be rationalized, we get

$$\begin{aligned} & u(1, \theta) + \imath v(1, \theta) \\ &= \frac{1}{2\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \mathrm{e}^{\imath k\Delta\theta} \, \left[u(1, \theta_1) + \imath v(1, \theta_1) \right] (-\imath) \, \frac{\mathrm{e}^{\imath\Delta\theta/2}}{\sin(\Delta\theta/2)} \\ &= \frac{1}{2\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \left[v(1, \theta_1) - \imath u(1, \theta_1) \right] \frac{\mathrm{e}^{\imath(k+1/2)\Delta\theta}}{\sin(\Delta\theta/2)} \\ &= \frac{1}{2\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \frac{\left[v(1, \theta_1) - \imath u(1, \theta_1) \right] \left[\cos(k_1\Delta\theta) + \imath \sin(k_1\Delta\theta) \right]}{\sin(\Delta\theta/2)}, \end{aligned} \tag{47}$$

where $\Delta\theta = \theta_1 - \theta$ and $k_1 = k + 1/2$, with $k \in \{1, 2, 3, \ldots, \infty\}$. Expanding the numerator in the integrand of this integral we have

$$\begin{aligned} & \left[v(1, \theta_1) - \imath u(1, \theta_1) \right] \left[\cos(k_1\Delta\theta) + \imath \sin(k_1\Delta\theta) \right] \\ &= \left[\sin(k_1\Delta\theta) u(1, \theta_1) + \cos(k_1\Delta\theta) v(1, \theta_1) \right] + \\ &\quad + \imath \left[\sin(k_1\Delta\theta) v(1, \theta_1) - \cos(k_1\Delta\theta) u(1, \theta_1) \right], \end{aligned} \tag{48}$$

and therefore we are left with

$$\begin{aligned} & u(1, \theta) + \imath v(1, \theta) \\ &= \frac{1}{2\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \frac{\sin(k_1\Delta\theta) u(1, \theta_1) + \cos(k_1\Delta\theta) v(1, \theta_1)}{\sin(\Delta\theta/2)} \\ &\quad + \imath \, \frac{1}{2\pi} \mathrm{PV}\int_{-\pi}^{\pi} d\theta_1 \, \frac{\sin(k_1\Delta\theta) v(1, \theta_1) - \cos(k_1\Delta\theta) u(1, \theta_1)}{\sin(\Delta\theta/2)}, \end{aligned} \tag{49}$$

where $\Delta\theta = \theta_1 - \theta$ and $k_1 = k + 1/2$, with $k \in \{1, 2, 3, \ldots, \infty\}$. Separating the real and imaginary parts we therefore obtain an infinite collection of identities in the form

$$\begin{aligned} & u(1,\theta) \\ = \; & \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\, \frac{\sin\!\left[(k+1/2)\Delta\theta\right] u(1,\theta_1) + \cos\!\left[(k+1/2)\Delta\theta\right] v(1,\theta_1)}{\sin(\Delta\theta/2)}, \\ & v(1,\theta) \\ = \; & \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\, \frac{\sin\!\left[(k+1/2)\Delta\theta\right] v(1,\theta_1) - \cos\!\left[(k+1/2)\Delta\theta\right] u(1,\theta_1)}{\sin(\Delta\theta/2)}, \end{aligned} \tag{50}$$

where $\Delta\theta = \theta_1 - \theta$ and $k \in \{1, 2, 3, \ldots, \infty\}$. Recalling now that $f(\theta) = u(1, \theta)$, and also that $g(\theta) = v(1, \theta)$, almost everywhere over the unit circle, one obtains the results in Equation (43), and therefore this completes the proof of Theorem 4.

Note that since this is an infinite collection of integral identities, satisfied by $f(\theta)$ and $g(\theta)$ for all strictly positive k, it follows that the right-hand sides of the equations above do not, in fact, depend on k. If one recognizes in the first term of each one of these two equations the well-known result for the k^{th} partial sums of the corresponding Fourier series in terms of Dirichlet integrals [3], then it follows that the other terms must be the corresponding remainders. This provides us with some level of understanding of the nature of this infinite set of identities. In the next section we will prove that one does obtain in fact the partial sums and remainders of the corresponding Fourier series directly from our complex-analytic structure.

6 Remainders of Fourier Series

We will now derive certain expressions for the partial sums and for the corresponding remainders of the Fourier series. In order to do this, let $f(\theta)$ be a zero-average integrable real function defined on $[-\pi, \pi]$ and let the real numbers $\alpha_0 = 0$, α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, be its Fourier coefficients. We then define the complex coefficients $c_0 = 0$ and c_k shown in Equation (5), and thus construct the corresponding proper inner analytic function $w(z)$ within the open unit disk, using the power series $S(z)$ given in Equation (6), which, as was shown in [1], always converges for $|z| < 1$. Considering that $c_0 \equiv 0$, the partial sums of the first N terms of this series are given by

$$S_N(z) = \sum_{k=0}^{N} c_k z^k, \tag{51}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$, a complex sequence for each value of z which, for $|z| < 1$, we already know to converge to $w(z)$ in the $N \to \infty$ limit. Note however that, since $S_N(z)$ is a polynomial of order N and therefore an analytic function over the whole complex plane, this expression itself can be consistently considered for all finite N and all z, and in particular for z on the unit circle, where $|z| = 1$. One can also define the corresponding remainders of the complex power series, in the usual way, as

$$R_N(z) = w(z) - S_N(z). \tag{52}$$

We will now prove the following theorem.

Theorem 5: *Given an arbitrary zero-average integrable real function $f(\theta)$ on the unit circle and the corresponding Fourier-conjugate real function $g(\theta)$, if $S_N^F(\theta) = \Re[S_N(1,\theta)]$ is the $N^{\rm th}$ partial sum of the Fourier series of $f(\theta)$, and if $R_N^F(\theta) = \Re[R_N(1,\theta)]$ is the corresponding remainder of that Fourier series, then we have that this partial sum and this remainder are given by the following integrals:*

$$\begin{aligned} S_N^F(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_1\, \frac{\sin\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\, f(\theta_1), \\ R_N^F(\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi} d\theta_1\, \frac{\cos\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\, g(\theta_1), \end{aligned} \tag{53}$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1, 2, 3, \ldots, \infty\}$.

Note that the integral in the expression of the partial sum is the known Dirichlet integral, while the one in the expression of the reminder is similar but not identical to it.

Proof 5.1:

In order to prove this theorem, let us consider the complex partial sums $S_N(z)$ as given in Equation (51). In addition to this, the complex coefficients c_k may be written as integrals involving $w(z)$, with the use of the Cauchy integral formulas,

$$c_k = \frac{1}{2\pi\imath}\oint_C dz\, \frac{w(z)}{z^{k+1}}, \tag{54}$$

for $k \in \{0, 1, 2, 3, \ldots, \infty\}$, where C can be taken as a circle centered at the origin, with radius $\rho \leq 1$. The reason why we may include the case $\rho = 1$ here is that, as was shown in [1], as a function of ρ the expression above for c_k is not only constant within the open unit disk, but also continuous from within at the unit circle. In this way the coefficients c_k may be written back in terms of the inner analytic function $w(z)$. If we substitute this expression for c_k back in the partial sums of the complex power series shown in Equation (51) we get

$$\begin{aligned} S_N(z) &= \sum_{k=0}^{N} z^k\, \frac{1}{2\pi\imath}\oint_C dz_1\, \frac{w(z_1)}{z_1^{k+1}} \\ &= \frac{1}{2\pi\imath}\oint_C dz_1\, \frac{w(z_1)}{z_1}\sum_{k=0}^{N}\left(\frac{z}{z_1}\right)^k, \end{aligned} \tag{55}$$

where z can have any value, but where we must have $|z_1| \leq 1$. The sum is now a finite geometric progression, so that we have its value in closed form,

$$\begin{aligned} S_N(z) &= \frac{1}{2\pi\imath}\oint_C dz_1\, \frac{w(z_1)}{z_1}\, \frac{1-(z/z_1)^{N+1}}{1-(z/z_1)} \\ &= \frac{1}{2\pi\imath}\oint_C dz_1\, \frac{w(z_1)}{z_1 - z} - \frac{z^{N+1}}{2\pi\imath}\oint_C dz_1\, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}. \end{aligned} \tag{56}$$

There are now two relevant cases to be considered here, the case in which $|z| < |z_1|$ and the case in which $|z| > |z_1|$. In the first case, since the explicit simple pole of the integrand at the position $z_1 = z$ lies within the integration contour, we have in the first term the Cauchy integral formula for $w(z)$, and therefore we get

$$S_N(z) = w(z) - \frac{z^{N+1}}{2\pi\imath} \oint_C dz_1 \, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}. \tag{57}$$

This is the equation that allows us to write an explicit expression for the remainder of the complex power series within the open unit disk, thus making it easier to discuss its convergence there. In the other case, in which $|z| > |z_1|$, the explicit simple pole of the integrand at the position z lies outside of the integration contour, and therefore by the Cauchy-Goursat theorem we just have zero in the first term, so that we get

$$S_N(z) = -\frac{z^{N+1}}{2\pi\imath} \oint_C dz_1 \, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}. \tag{58}$$

This provides us, therefore, with an explicit expression for the partial sums, but not for the remainder. The only other possible case is that in which $|z| = |z_1|$, in which both z_1 and z are over the circle C of radius ρ_1, and therefore so is the explicit simple pole of the integrand at the position $z_1 = z$. In this case, just as we did before in Section 3, we may slightly deform the integration contour C in order to have it pass on one side or the other of the simple pole of the integrand at $z_1 = z$. If we use a deformed contour $C_\ominus$ that *excludes* the pole from its interior, then we have, instead of Equation (57),

$$S_N(z) = 0 - \frac{z^{N+1}}{2\pi\imath} \oint_{C_\ominus} dz_1 \, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}, \tag{59}$$

while if we use a deformed contour $C_\oplus$ that *includes* the pole in its interior, then we have, just as in Equation (57),

$$S_N(z) = w(z) - \frac{z^{N+1}}{2\pi\imath} \oint_{C_\oplus} dz_1 \, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}. \tag{60}$$

Once more, since by the Sokhotskii-Plemelj theorem [13] the Cauchy principal value of the integral over C is the arithmetic average of these two integrals, taking the average of Equations (59) and (60) we obtain the expression

$$S_N(z) = \frac{w(z)}{2} - \frac{z^{N+1}}{2\pi\imath} \, \mathrm{PV}\!\oint_C dz_1 \, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}, \tag{61}$$

where both z_1 and z are now on the circle C of radius ρ_1. Since we have that the corresponding remainder of the series is defined as given in Equation (52), we get a corresponding expression for the remainder, in terms of the same integral,

$$R_N(z) = \frac{w(z)}{2} + \frac{z^{N+1}}{2\pi\imath} \, \mathrm{PV}\!\oint_C dz_1 \, \frac{w(z_1)}{z_1^{N+1}(z_1 - z)}, \tag{62}$$

where both z_1 and z are on the circle C of radius ρ_1. We have therefore the pair of equations

$$\begin{aligned} S_N(z) &= \frac{w(z)}{2} - I_N(z), \\ R_N(z) &= \frac{w(z)}{2} + I_N(z), \end{aligned} \tag{63}$$

and we must now write the integral $I_N(z)$ explicitly in terms of ρ_1, θ_1 and θ,

$$
\begin{aligned}
I_N(z) & = \frac{z^{N+1}}{2\pi\imath}\,\mathrm{PV}\oint_C dz_1\,\frac{w(z_1)}{z_1^{N+1}(z_1-z)}\\
& = \frac{\rho_1^{N+1}\,\mathrm{e}^{\imath(N+1)\theta}}{2\pi\imath}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\imath\rho_1\,\mathrm{e}^{\imath\theta_1}\,\frac{u(\rho_1,\theta_1)+\imath v(\rho_1,\theta_1)}{\rho_1^{N+1}\,\mathrm{e}^{\imath(N+1)\theta_1}\,(\rho_1\,\mathrm{e}^{\imath\theta_1}-\rho_1\,\mathrm{e}^{\imath\theta})}\\
& = \frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\mathrm{e}^{-\imath(N+1)\Delta\theta}\,\frac{u(\rho_1,\theta_1)+\imath v(\rho_1,\theta_1)}{1-\mathrm{e}^{-\imath\Delta\theta}},
\end{aligned}
\tag{64}
$$

where $\Delta\theta=\theta_1-\theta$ and $N\in\{1,2,3,\ldots,\infty\}$. Once again we must rationalize the integrand, and using once more the result shown in Equation (22) we get

$$
\begin{aligned}
& I_N(\rho_1,\theta)\\
& = \frac{1}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\mathrm{e}^{-\imath(N+1)\Delta\theta}\left[u(\rho_1,\theta_1)+\imath v(\rho_1,\theta_1)\right](-\imath)\frac{\mathrm{e}^{\imath\Delta\theta/2}}{\sin(\Delta\theta/2)}\\
& = \frac{1}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\left[v(\rho_1,\theta_1)-\imath u(\rho_1,\theta_1)\right]\frac{\mathrm{e}^{-\imath(N+1/2)\Delta\theta}}{\sin(\Delta\theta/2)}\\
& = \frac{1}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\frac{\left[v(\rho_1,\theta_1)-\imath u(\rho_1,\theta_1)\right]\left[\cos(N_1\Delta\theta)-\imath\sin(N_1\Delta\theta)\right]}{\sin(\Delta\theta/2)},
\end{aligned}
\tag{65}
$$

where $\Delta\theta=\theta_1-\theta$ and $N_1=N+1/2$, with $N\in\{1,2,3,\ldots,\infty\}$. Expanding the numerator in the integrand of this integral we have

$$
\begin{aligned}
& \left[v(\rho_1,\theta_1)-\imath u(\rho_1,\theta_1)\right]\left[\cos(N_1\Delta\theta)-\imath\sin(N_1\Delta\theta)\right]\\
& \quad = -\left[\sin(N_1\Delta\theta)u(\rho_1,\theta_1)-\cos(N_1\Delta\theta)v(\rho_1,\theta_1)\right]+\\
& \qquad -\imath\left[\sin(N_1\Delta\theta)v(\rho_1,\theta_1)+\cos(N_1\Delta\theta)u(\rho_1,\theta_1)\right],
\end{aligned}
\tag{66}
$$

and therefore we are left with the following expression for our integral,

$$
\begin{aligned}
& I_N(\rho_1,\theta)\\
& = -\frac{1}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin(N_1\Delta\theta)u(\rho_1,\theta_1)-\cos(N_1\Delta\theta)v(\rho_1,\theta_1)}{\sin(\Delta\theta/2)}+\\
& \quad -\frac{\imath}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin(N_1\Delta\theta)v(\rho_1,\theta_1)+\cos(N_1\Delta\theta)u(\rho_1,\theta_1)}{\sin(\Delta\theta/2)},
\end{aligned}
\tag{67}
$$

where $\Delta\theta=\theta_1-\theta$ and $N_1=N+1/2$, with $N\in\{1,2,3,\ldots,\infty\}$. Once again we note that, since by construction the real and imaginary parts $u(\rho_1,\theta)$ and $v(\rho_1,\theta)$ of $w(\rho_1,\theta)$ for $\rho_1=1$ are integrable real functions on the unit circle, and since there are no other dependencies on ρ_1 in this equation, we may now take the $\rho_1\to 1_{(-)}$ limit of this expression, in which the principal value acquires its usual real meaning on the unit circle, thus obtaining

$$
\begin{aligned}
& I_N(1,\theta)\\
& = -\frac{1}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\left[(N+1/2)\Delta\theta\right]u(1,\theta_1)-\cos\left[(N+1/2)\Delta\theta\right]v(1,\theta_1)}{\sin(\Delta\theta/2)}+\\
& \quad -\frac{\imath}{4\pi}\,\mathrm{PV}\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\left[(N+1/2)\Delta\theta\right]v(1,\theta_1)+\cos\left[(N+1/2)\Delta\theta\right]u(1,\theta_1)}{\sin(\Delta\theta/2)},
\end{aligned}
\tag{68}
$$

where $\Delta\theta=\theta_1-\theta$ and $N\in\{1,2,3,\ldots,\infty\}$, in terms of which we now have the pair of equations at the unit circle

$$\begin{aligned}S_N(1,\theta) &= \frac{w(1,\theta)}{2} - I_N(1,\theta),\\ R_N(1,\theta) &= \frac{w(1,\theta)}{2} + I_N(1,\theta).\end{aligned} \tag{69}$$

Using now the infinite collection of identities in Equation (50) for the case $k = N$, which allow us to write $w(1,\theta)/2$ in terms of integrals similar to those in $I_N(1,\theta)$,

$$\begin{aligned}&\frac{w(1,\theta)}{2}\\ &= \frac{1}{4\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)+\cos\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)}{\sin(\Delta\theta/2)}+\\ &\quad+\frac{\imath}{4\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)-\cos\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)}{\sin(\Delta\theta/2)},\end{aligned} \tag{70}$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1,2,3,\ldots,\infty\}$, we may write for the complex partial sums

$$\begin{aligned}&S_N(1,\theta)\\ &= \frac{w(1,\theta)}{2} - I_N(1,\theta)\\ &= \frac{1}{4\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)+\cos\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)}{\sin(\Delta\theta/2)}+\\ &\quad+\frac{\imath}{4\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)-\cos\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)}{\sin(\Delta\theta/2)}+\\ &\quad+\frac{1}{4\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)-\cos\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)}{\sin(\Delta\theta/2)}+\\ &\quad+\frac{\imath}{4\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)+\cos\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)}{\sin(\Delta\theta/2)}.\end{aligned} \tag{71}$$

As one can see in this equation, all the terms involving $\cos\big[(N+1/2)\Delta\theta\big]$ cancel off, and therefore we are left with

$$\begin{aligned}S_N(1,\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)}{\sin(\Delta\theta/2)}+\\ &\quad+\frac{\imath}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)}{\sin(\Delta\theta/2)},\end{aligned} \tag{72}$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1,2,3,\ldots,\infty\}$. We now observe that, since by hypothesis $f(\theta)$ is integrable on the unit circle, the Cauchy principal value refers only to the possible explicit non-integrable singularity of the integrands, due to the zero of the denominators at $\theta_1 = \theta$. However, since the numerators of the integrands are also zero at that point, the integrands are not really divergent at all at that point, so that from this point on we may drop the principal value. We have therefore our final results for the real partial sums, for both $u(1,\theta)$ and $v(1,\theta)$,

$$\begin{aligned}S_N^{F,u}(\theta) &= \Re[S_N(1,\theta)]\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,u(1,\theta_1),\\ S_N^{F,v}(\theta) &= \Im[S_N(1,\theta)]\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_1\,\frac{\sin\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,v(1,\theta_1),\end{aligned} \tag{73}$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1, 2, 3, \ldots, \infty\}$. These are the well-known results for the partial sums, in terms of Dirichlet integrals [3]. Note that the two equations above have exactly the same form, which is to be expected, since the result holds for all zero-average integrable real functions, including of course both $u(1, \theta_1)$ and $v(1, \theta_1)$. Therefore, given an arbitrary zero-average integrable real function $f(\theta)$ on the unit circle, we have that the partial sums of its Fourier series are given by

$$S_N^F(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_1 \, \frac{\sin\left[(N + 1/2)(\theta_1 - \theta)\right]}{\sin\left[(\theta_1 - \theta)/2\right]} \, f(\theta_1), \tag{74}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$. Note that, although this result is already very well known, we have showed here that it does follow from our complex-analytic structure. This completes the proof of the first part of Theorem 5.

Once more, it is interesting to observe that this relation can be interpreted as a linear integral operator acting on the space of zero-average integrable real functions defined on the unit circle, this time resulting in the N^{th} partial sum of the Fourier series of a zero-average integrable real function, a partial sum which is itself a zero-average integrable real function. The integration kernel of this integral operator $\mathcal{D}_{\mathrm{s}}[N, f(\theta)]$ depends only on N and on the difference $\theta - \theta_1$, and is given by

$$K_{\mathcal{D}_{\mathrm{s}}}(N, \theta - \theta_1) = \frac{1}{2\pi} \, \frac{\sin\left[(N + 1/2)(\theta_1 - \theta)\right]}{\sin\left[(\theta_1 - \theta)/2\right]}, \tag{75}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$, so that the action of the operator on $f(\theta)$ can be written as

$$\mathcal{D}_{\mathrm{s}}[N, f(\theta)] = \int_{-\pi}^{\pi} d\theta_1 \, K_{\mathcal{D}_{\mathrm{s}}}(N, \theta - \theta_1) f(\theta_1). \tag{76}$$

Considering that its kernel is given by a Dirichlet integral, one might call this the *Dirichlet operator*, so that the N^{th} partial sum of the Fourier series of $f(\theta)$ is given by the action of this operator on the zero-average integrable real function $f(\theta)$,

$$S_N^F(\theta) = \mathcal{D}_{\mathrm{s}}[N, f(\theta)], \tag{77}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$. Note that $\mathcal{D}_{\mathrm{s}}[N, f(\theta)]$ constitutes in fact a whole collection of linear integral operators acting on the space of zero-average integrable real functions.

Proof 5.2:

Using once more the very same elements that were used above for the complex partial sums, we may also write corresponding results for the complex remainders,

$$\begin{aligned}
& R_N(1, \theta) \\
& \quad = \quad \frac{w(1, \theta)}{2} + I_N(1, \theta) \\
& \quad = \quad \frac{1}{4\pi} \, \mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1 \, \frac{\sin\left[(N + 1/2)\Delta\theta\right] u(1, \theta_1) + \cos\left[(N + 1/2)\Delta\theta\right] v(1, \theta_1)}{\sin(\Delta\theta/2)} + \\
& \qquad\quad + \frac{\imath}{4\pi} \, \mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1 \, \frac{\sin\left[(N + 1/2)\Delta\theta\right] v(1, \theta_1) - \cos\left[(N + 1/2)\Delta\theta\right] u(1, \theta_1)}{\sin(\Delta\theta/2)} + \\
& \qquad\quad - \frac{1}{4\pi} \, \mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1 \, \frac{\sin\left[(N + 1/2)\Delta\theta\right] u(1, \theta_1) - \cos\left[(N + 1/2)\Delta\theta\right] v(1, \theta_1)}{\sin(\Delta\theta/2)} +
\end{aligned}$$

$$- \frac{\imath}{4\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\sin\!\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)+\cos\!\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)}{\sin(\Delta\theta/2)}. \qquad (78)$$

As one can see in this equation, this time all the terms involving $\sin\big[(N+1/2)\Delta\theta\big]$ chancel off, and therefore we are left with

$$\begin{aligned} R_N(1,\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]\,v(1,\theta_1)}{\sin(\Delta\theta/2)}+ \\ &\quad -\frac{\imath}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]\,u(1,\theta_1)}{\sin(\Delta\theta/2)}, \end{aligned} \qquad (79)$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1,2,3,\ldots,\infty\}$. We have therefore our results for the real remainders, for both $u(1,\theta)$ and $v(1,\theta)$,

$$\begin{aligned} R_N^{F,u}(\theta) &= \Re[R_N(1,\theta)] \\ &= \frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,v(1,\theta_1), \\ R_N^{F,v}(\theta) &= \Im[R_N(1,\theta)] \\ &= -\frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,u(1,\theta_1), \end{aligned} \qquad (80)$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1,2,3,\ldots,\infty\}$. Recalling that $f(\theta) = u(1,\theta)$ and that $g(\theta) = v(1,\theta)$ almost everywhere over the unit circle, this completes the proof of Theorem 5.

We believe that these are new results, written in terms of integrals which are similar to the Dirichlet integrals, but not identical to them. Note that the remainder of the series of $f(\theta)$ is given as an integral involving its Fourier-conjugate function $g(\theta)$, and vice versa. Therefore, we conclude that the convergence condition of the Fourier series of a given real function does not depend directly on that function, but only indirectly, through the properties of its Fourier-conjugate real function.

Since we know that these two real functions are related by the compact Hilbert transform, we may write these equations as

$$\begin{aligned} R_N^{F,u}(\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,\mathcal{H}_{\rm c}[u(1,\theta_1)], \\ R_N^{F,v}(\theta) &= \frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,\mathcal{H}_{\rm c}[v(1,\theta_1)], \end{aligned} \qquad (81)$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1,2,3,\ldots,\infty\}$. Note that the two results are now identical in form. Therefore, given an arbitrary zero-average integrable real function $f(\theta)$ on the unit circle, we have our final result for the remainder of its Fourier series,

$$R_N^{F}(\theta) = \frac{1}{2\pi}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta_1\,\frac{\cos\!\big[(N+1/2)\Delta\theta\big]}{\sin(\Delta\theta/2)}\,\mathcal{H}_{\rm c}[f(\theta_1)], \qquad (82)$$

where $\Delta\theta = \theta_1 - \theta$ and $N \in \{1,2,3,\ldots,\infty\}$.

Once again, it is interesting to observe that this relation can be interpreted as a linear integral operator acting on the space of integrable zero-average real functions defined on the unit circle. The operator $\mathcal{D}_{\rm c}[N,g(\theta)]$, acting on the compact Hilbert transform $g(\theta)$ of such a function, results in the $N^{\rm th}$ remainder of the Fourier series of the original function

$f(\theta)$, a remainder which, if it exists at all, is itself a zero-average integrable real function. The integration kernel of the integral operator depends only on N and on the difference $\theta - \theta_1$, and is given by

$$K_{\mathcal{D}_c}(N, \theta - \theta_1) = \frac{1}{2\pi} \frac{\cos\left[(N + 1/2)(\theta_1 - \theta)\right]}{\sin\left[(\theta_1 - \theta)/2\right]}, \tag{83}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$, so that the action of the operator on an arbitrarily given zero-average integrable real function $g(\theta)$ can be written as

$$\mathcal{D}_c[N, g(\theta)] = \mathrm{PV}\!\int_{-\pi}^{\pi} d\theta_1\, K_{\mathcal{D}_c}(N, \theta - \theta_1) g(\theta_1). \tag{84}$$

This new operator, which we might refer to as the *conjugate Dirichlet operator*, is similar to the Dirichlet operator, and is such that the N^{th} remainder of the Fourier series of the real function $f(\theta)$ is given by the action of this operator on the Fourier-conjugate function $g(\theta)$ of the real function $f(\theta)$,

$$R_N^F(\theta) = \mathcal{D}_c[N, g(\theta)], \tag{85}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$. Note once more that $\mathcal{D}_c[N, g(\theta)]$ constitutes in fact a whole collection of linear integral operators acting on the space of zero-average integrable real functions. Note also that, since by hypothesis $f(\theta)$ and $g(\theta)$ are integrable on the unit circle, the Cauchy principal value refers only to the explicit non-integrable singularity of the integration kernel at $\theta_1 = \theta$. In this operator notation we have therefore that the remainder of the Fourier series of an arbitrarily given zero-average integrable real function $f(\theta)$ is given by the composition of $\mathcal{D}_c[N, g(\theta)]$ with $\mathcal{H}_c[f(\theta)]$,

$$R_N^F(\theta) = \mathcal{D}_c\left[N, \mathcal{H}_c[f(\theta)]\right]. \tag{86}$$

Note that, according to the inverse of the relation shown in Equation (42) we may write as well that

$$R_N^F(\theta) = \mathcal{H}_c^{-1}\left[\mathcal{D}_c[N, f(\theta)]\right], \tag{87}$$

which constitutes an equivalent way to express the remainder of the Fourier series of $f(\theta)$ in terms of the function itself.

Finally note that, given the results obtained here for the partial sums and remainders of the Fourier series, the expressions in the infinite collection of identities shown in Equation (50) have now, in fact, the very simple interpretation that was alluded to there, since we now see that they can in fact be written as

$$\begin{aligned} f(\theta) &= S_N^{F,f}(\theta) + R_N^{F,f}(\theta), \\ g(\theta) &= S_N^{F,g}(\theta) + R_N^{F,g}(\theta), \end{aligned} \tag{88}$$

where $N \in \{1, 2, 3, \ldots, \infty\}$, a fact which greatly clarifies the nature of that infinite collections of identities.

7 The Convergence Condition

Given an arbitrary zero-average integrable real function $f(\theta)$ defined on the unit circle, the necessary and sufficient condition for the convergence of its Fourier series at the point θ is stated very simply as the condition that

$$\lim_{N\to\infty} R_N^F(\theta) = 0, \tag{89}$$

where $R_N^F(\theta)$ is the remainder of that Fourier series, as given in Section 6. According to what was shown in that Section, in terms of the integral operator $\mathcal{D}_{\rm c}[N, g(\theta)]$ this translates therefore as the condition that

$$\lim_{N\to\infty} \mathcal{D}_{\rm c}[N, g(\theta)] = 0, \tag{90}$$

where $g(\theta)$ is the Fourier-conjugate function of $f(\theta)$, which is given by the compact Hilbert transform $\mathcal{H}_{\rm c}[f(\theta)]$, leading therefore to the composition of the two operators,

$$\lim_{N\to\infty} \mathcal{D}_{\rm c}\big[N, \mathcal{H}_{\rm c}[f(\theta)]\big] = 0. \tag{91}$$

Equivalently, we may define the linear integral operator $\mathcal{D}_{\rm r}[N, f(\theta)]$ to be this composition of $\mathcal{D}_{\rm c}[N, f(\theta)]$ with $\mathcal{H}_{\rm c}[f(\theta)]$,

$$\mathcal{D}_{\rm r}[N, f(\theta)] = \mathcal{D}_{\rm c}\big[N, \mathcal{H}_{\rm c}[f(\theta)]\big], \tag{92}$$

that therefore maps $f(\theta)$ directly onto the $N^{\rm th}$ remainder of its Fourier series,

$$R_N^F(\theta) = \mathcal{D}_{\rm r}[N, f(\theta)], \tag{93}$$

so that the convergence condition of the Fourier series of $f(\theta)$ can now be written as

$$\lim_{N\to\infty} \mathcal{D}_{\rm r}[N, f(\theta)] = 0. \tag{94}$$

Combining the integration kernels of the operators $\mathcal{D}_{\rm c}[N, f(\theta)]$, given in Equation (83), and $\mathcal{H}_{\rm c}[f(\theta)]$, given in Equation (27), we may write an integration kernel for the operator $\mathcal{D}_{\rm r}[N, f(\theta)]$, which is not given explicitly, but rather remains expressed as an integral over the unit circle,

$$K_{\mathcal{D}_{\rm r}}(N, \theta, \theta_1) = \frac{1}{4\pi^2}\,{\rm PV}\!\!\int_{-\pi}^{\pi} d\theta'\,\frac{\cos\big[(N+1/2)(\theta'-\theta)\big]\cos\big[(\theta'-\theta_1)/2\big]}{\sin\big[(\theta'-\theta)/2\big]\sin\big[(\theta'-\theta_1)/2\big]}, \tag{95}$$

in terms of which the action of the operator $\mathcal{D}_{\rm r}[N, f(\theta)]$ on $f(\theta)$ is given by

$$\mathcal{D}_{\rm r}[N, f(\theta)] = {\rm PV}\!\!\int_{-\pi}^{\pi} d\theta_1\, K_{\mathcal{D}_{\rm r}}(N, \theta, \theta_1) f(\theta_1). \tag{96}$$

Note that at this point it is not clear whether or not the integration kernel depends only on the difference $\theta - \theta_1$. We will prove that it does, and we will also write it in a somewhat more convenient form. In this section we will prove the following theorem.

Theorem 6: *Given an arbitrary zero-average integrable real function $f(\theta)$ defined on the unit circle, the necessary and sufficient condition for the convergence of its Fourier series at the point θ is as follows:*

$$\lim_{N\to\infty} {\rm PV}\!\!\int_{-\pi}^{\pi} d\theta_1\, K_{\mathcal{D}_{\rm r}}(N, \theta-\theta_1) f(\theta_1) = 0, \tag{97}$$

where the integration kernel is given by

$$
\begin{aligned}
&K_{\mathcal{D}_{\rm r}}(N,\theta-\theta_1)\\
&= \frac{\cos\bigl[(N+1)(\theta-\theta_1)/2\bigr]}{4\pi^2}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta'\,\frac{\cos(N\theta')}{\cos\bigl[(\theta-\theta_1)/2\bigr]-\cos(\theta')}+\\
&\qquad +\frac{\cos\bigl[N(\theta-\theta_1)/2\bigr]}{4\pi^2}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta'\,\frac{\cos\bigl[(N+1)\theta'\bigr]}{\cos\bigl[(\theta-\theta_1)/2\bigr]-\cos(\theta')}.
\end{aligned}
\tag{98}
$$

Note that the two integrals in this form of the condition are almost identical, differing only by the exchange of N for $N+1$. Note also that the integrands of these integrals are singular at the points $\theta'=\pm(\theta-\theta_1)/2$. Note, finally, that the condition in Equation (97) means that the remainder $R_N^F(\theta)$ must exist, being a finite number for each N, as well as that its $N\to\infty$ limit must be zero. The existence of the remainder is, of course, equivalent to the existence of the integrals involved.

Proof 6.1:

We start by making in the integral in Equation (95) the transformation of variables

$$
\begin{aligned}
\theta'' &= \theta'-\frac{\theta+\theta_1}{2} \quad\Rightarrow\\
\theta' &= \theta''+\frac{\theta+\theta_1}{2},
\end{aligned}
\tag{99}
$$

which implies that

$$
\begin{aligned}
\theta'-\theta &= \theta''-\frac{\theta-\theta_1}{2},\\
\theta'-\theta_1 &= \theta''+\frac{\theta-\theta_1}{2},
\end{aligned}
\tag{100}
$$

and which also implies that $d\theta'=d\theta''$, so that we have

$$
\begin{aligned}
&K_{\mathcal{D}_{\rm r}}(N,\theta,\theta_1)\\
&= \frac{1}{4\pi^2}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta''\,\frac{\cos\bigl[N_1\theta''-N_1(\theta-\theta_1)/2\bigr]\cos\bigl[\theta''/2+(\theta-\theta_1)/4\bigr]}{\sin\bigl[\theta''/2-(\theta-\theta_1)/4\bigr]\sin\bigl[\theta''/2+(\theta-\theta_1)/4\bigr]},
\end{aligned}
\tag{101}
$$

where $N_1=N+1/2$ with $N\in\{1,2,3,\ldots,\infty\}$, and where we do not have to change the integration limits since the integration runs over a circle. Note that at this point it is already clearly apparent that $K_{\mathcal{D}_{\rm r}}(N,\theta,\theta_1)$ depends only on N and on the difference $\theta-\theta_1$, and therefore from now on we will write it as $K_{\mathcal{D}_{\rm r}}(N,\theta-\theta_1)$. Changing θ'' back to θ' and using the notation $\gamma=(\theta-\theta_1)/2$ we have

$$
\begin{aligned}
K_{\mathcal{D}_{\rm r}}(N,\theta-\theta_1) &= \frac{1}{4\pi^2}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta'\,\frac{\cos\bigl[N_1(\theta'-\gamma)\bigr]\cos\bigl[(\theta'+\gamma)/2\bigr]}{\sin\bigl[(\theta'-\gamma)/2\bigr]\sin\bigl[(\theta'+\gamma)/2\bigr]}\\
&= \frac{1}{4\pi^2}\,\mathrm{PV}\!\!\int_{-\pi}^{\pi}d\theta'\,\frac{P(N,\theta',\gamma)}{Q(\theta',\gamma)},
\end{aligned}
\tag{102}
$$

where $\gamma=(\theta-\theta_1)/2$ and $N_1=N+1/2$ with $N\in\{1,2,3,\ldots,\infty\}$. We will now manipulate the denominator $Q(\theta',\gamma)$ and the numerator $P(N,\theta',\gamma)$ in this integrand using trigonometric identities. We start with the denominator, and using the trigonometric identities for the sum of two angles we get

$$\begin{aligned}
Q(\theta',\gamma) &= \sin\big[(\theta'-\gamma)/2\big]\sin\big[(\theta'+\gamma)/2\big]\\
&= \big[\sin(\theta'/2)\cos(\gamma/2)-\cos(\theta'/2)\sin(\gamma/2)\big]\times\\
&\quad\times\big[\sin(\theta'/2)\cos(\gamma/2)+\cos(\theta'/2)\sin(\gamma/2)\big]\\
&= \sin^2(\theta'/2)\cos^2(\gamma/2)-\cos^2(\theta'/2)\sin^2(\gamma/2).
\end{aligned}\tag{103}$$

If we now write the cosines in terms of the corresponding sines we have

$$\begin{aligned}
Q(\theta',\gamma) &= \sin^2(\theta'/2)-\sin^2(\theta'/2)\sin^2(\gamma/2)+\\
&\quad-\sin^2(\gamma/2)+\sin^2(\theta'/2)\sin^2(\gamma/2)\\
&= \sin^2(\theta'/2)-\sin^2(\gamma/2).
\end{aligned}\tag{104}$$

Using now the half-angle trigonometric identities we finally have

$$\begin{aligned}
Q(\theta',\gamma) &= \frac{1-\cos(\theta')}{2}-\frac{1-\cos(\gamma)}{2}\\
&= \frac{\cos(\gamma)-\cos(\theta')}{2}.
\end{aligned}\tag{105}$$

It is important to note that this is an *even* function of θ'. Turning now to the numerator $P(N,\theta',\gamma)$, and using the trigonometric identities for the sum of two angles we get

$$\begin{aligned}
P(N,\theta',\gamma) &= \cos\big[N_1(\theta'-\gamma)\big]\cos\big[(\theta'+\gamma)/2\big]\\
&= \big[\cos(N_1\theta')\cos(N_1\gamma)+\sin(N_1\theta')\sin(N_1\gamma)\big]\times\\
&\quad\times\big[\cos(\theta'/2)\cos(\gamma/2)-\sin(\theta'/2)\sin(\gamma/2)\big]\\
&= \cos(N_1\theta')\cos(N_1\gamma)\cos(\theta'/2)\cos(\gamma/2)+\\
&\quad-\cos(N_1\theta')\cos(N_1\gamma)\sin(\theta'/2)\sin(\gamma/2)+\\
&\quad+\sin(N_1\theta')\sin(N_1\gamma)\cos(\theta'/2)\cos(\gamma/2)+\\
&\quad-\sin(N_1\theta')\sin(N_1\gamma)\sin(\theta'/2)\sin(\gamma/2).
\end{aligned}\tag{106}$$

Of these four terms, the first and last ones are even on θ', and the two middle ones are odd. Since the denominator is even and the integral on θ' shown in Equation (102) is over a symmetric interval, the integrals of the two middle terms will be zero, and therefore we can ignore these two terms of the numerator. We thus obtain for our kernel

$$K_{\mathcal{D}_{\rm r}}(N,\theta-\theta_1)=\frac{1}{4\pi^2}\,\mathrm{PV}\!\int_{-\pi}^{\pi}d\theta'\,\frac{T(N,\theta',\gamma)}{Q(\theta',\gamma)},\tag{107}$$

where the new numerator is given by

$$\begin{aligned}
T(N,\theta',\gamma) &= \Big\{\cos\big[(N+1/2)\theta'\big]\cos(\theta'/2)\Big\}\cos(N_1\gamma)\cos(\gamma/2)+\\
&\quad-\Big\{\sin\big[(N+1/2)\theta'\big]\sin(\theta'/2)\Big\}\sin(N_1\gamma)\sin(\gamma/2),
\end{aligned}\tag{108}$$

where $\gamma=(\theta-\theta_1)/2$ and $N_1=N+1/2$ with $N\in\{1,2,3,\ldots,\infty\}$. Using once again the trigonometric identities for the sum of two angles in order to work on the two expressions within pairs of curly brackets, we get

$$\begin{aligned}
B_{\rm c} &= \Big\{\cos\big[(N+1/2)\theta'\big]\cos(\theta'/2)\Big\}\\
&= \big[\cos(N\theta')\cos(\theta'/2)-\sin(N\theta')\sin(\theta'/2)\big]\cos(\theta'/2)\\
&= \cos(N\theta')\cos^2(\theta'/2)-\sin(N\theta')\sin(\theta'/2)\cos(\theta'/2),
\end{aligned}$$

$$
\begin{aligned}
B_{\rm s} &= \Big\{ \sin\big[(N+1/2)\theta'\big] \sin(\theta'/2) \Big\} \\
&= \big[\sin(N\theta') \cos(\theta'/2) + \cos(N\theta') \sin(\theta'/2)\big] \sin(\theta'/2) \\
&= \sin(N\theta') \cos(\theta'/2) \sin(\theta'/2) + \cos(N\theta') \sin^2(\theta'/2). \qquad (109)
\end{aligned}
$$

Using now the half-angle trigonometric identities in order to eliminate the functions $\cos(\theta'/2)$ and $\sin(\theta'/2)$ in favor of $\cos(\theta')$ and $\sin(\theta')$, we get

$$
\begin{aligned}
B_{\rm c} &= \cos(N\theta') \frac{1+\cos(\theta')}{2} - \sin(N\theta') \frac{\sin(\theta')}{2} \\
&= \frac{\cos(N\theta')}{2} + \frac{\cos(N\theta') \cos(\theta') - \sin(N\theta') \sin(\theta')}{2}, \\
B_{\rm s} &= \sin(N\theta') \frac{\sin(\theta')}{2} + \cos(N\theta') \frac{1-\cos(\theta')}{2} \\
&= \frac{\cos(N\theta')}{2} - \frac{\cos(N\theta') \cos(\theta') - \sin(N\theta') \sin(\theta')}{2}. \qquad (110)
\end{aligned}
$$

Using once more the trigonometric identities for the sum of two angles we get

$$
\begin{aligned}
B_{\rm c} &= \frac{\cos(N\theta')}{2} + \frac{\cos\big[(N+1)\theta'\big]}{2}, \\
B_{\rm s} &= \frac{\cos(N\theta')}{2} - \frac{\cos\big[(N+1)\theta'\big]}{2}. \qquad (111)
\end{aligned}
$$

We get therefore for our new numerator, using yet gain the trigonometric identities for the sum of two angles,

$$
\begin{aligned}
T(N,\theta',\gamma) &= \frac{\cos(N\theta')}{2} \big[\cos(N_1\gamma) \cos(\gamma/2) - \sin(N_1\gamma) \sin(\gamma/2)\big] + \\
&\quad + \frac{\cos\big[(N+1)\theta'\big]}{2} \big[\cos(N_1\gamma) \cos(\gamma/2) + \sin(N_1\gamma) \sin(\gamma/2)\big] \\
&= \frac{\cos(N\theta') \cos\big[(N+1)\gamma\big]}{2} + \frac{\cos\big[(N+1)\theta'\big] \cos(N\gamma)}{2}. \qquad (112)
\end{aligned}
$$

We therefore have for the kernel

$$
\begin{aligned}
& K_{\mathcal{D}_{\rm r}}(N,\theta-\theta_1) \\
&= \frac{1}{4\pi^2} \,\mathrm{PV}\!\int_{-\pi}^{\pi} d\theta' \, \frac{\cos(N\theta') \cos\big[(N+1)\gamma\big] + \cos\big[(N+1)\theta'\big] \cos(N\gamma)}{\cos(\gamma) - \cos(\theta')}, \qquad (113)
\end{aligned}
$$

where $\gamma = (\theta-\theta_1)/2$ and $N \in \{1,2,3,\ldots,\infty\}$. This is exactly the form of the integration kernel of the operator $\mathcal{D}_{\rm r}[N,f(\theta)]$ given in Equation (98), so that this completes the proof of Theorem 6.

Since the condition stated in Theorem 6 is a necessary and sufficient condition on the real function $f(\theta)$ for the convergence of its Fourier series, any other such condition must be equivalent to it. Note that this type of condition is not really what is usually meant by a Fourier theorem. Those are just sufficient conditions for the convergence, usually related to some fairly simple and easily identifiable characteristics of the real functions, such as continuity, differentiability, existence of lateral limits, or limited variation. However, all such Fourier theorems must be related to this condition in the sense that they must imply its validity.

8 Conclusions and Outlook

We have shown that the complex-analytic structure that we introduced in [1] can be used to discuss the issue of the convergence of Fourier series. Using that structure we derived the known formulas for the partial sums of a Fourier series, in terms of Dirichlet integrals. From that same structure, and in fact as part of the same argument, we also obtained a new result, namely formulas giving the remainders of a Fourier series in terms of a similar but considerably more complex type of integral, in fact a double integral.

The introduction of a modified version of the Hilbert transform, which we named the compact Hilbert transform, had a central role to play in this development. The main result that follows from it is the necessary and sufficient condition for the convergence of a Fourier series, which is expressed in Equations (97) and (98). One might consider whether or not the integration kernel involved in this new type of Dirichlet integral, versions of which are shown in Equations (95) and (98), can be cast in some more convenient form, and possibly even calculated in close form in some useful way. So far no meaningful results of this type have been found.

A direct calculation of this integration kernel in closed form seems to be difficult, and possibly not overly useful, since it seems that such calculations tend to just take us back to the rather trivial identity

$$\mathcal{D}_{\mathrm{r}}[N, f(\theta)] = \mathcal{I}[f(\theta)] - \mathcal{D}_{\mathrm{s}}[N, f(\theta)], \tag{114}$$

where $\mathcal{I}[f(\theta)]$ is the identity operator, whose integration kernel is a Dirac delta "function", an identity which is simply equivalent to

$$R_N^F(\theta) = f(\theta) - S_N^F(\theta). \tag{115}$$

In so far as can be currently ascertained, this identity does not provide any constructively useful information about the convergence problem.

We already knew that he convergence problem of Fourier series relates to the existence and nature of singularities of the corresponding inner analytic functions on the unit circle. The convergence condition expressed in Equations (97) and (98) reflect this relation, since the existence of non-integrable singularities at the unit circle may very well disturb the validity of the condition by making the integrals diverge or at least not go to zero in the $N \to \infty$ limit. Note that this relation is non-local because, due to the fact that the integrals are over the whole unit circle, the existence of a non-integrable singularity at a single point may prevent the convergence of the series at almost all points.

It seems to us, at this time, that the most promising possible development of the convergence analysis presented here is probably one targeted at detailing the relation between the convergence of the series and the specific classification of the singularities on the unit circle, involving the concepts of hard and soft singularities, as well as the corresponding degrees of hardness or softness that can be attributed to them.

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