Complex Analysis of Real Functions

V: The Dirichlet Problem on the Plane

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Abstract

In the context of the correspondence between real functions on the unit circle and inner analytic functions within the open unit disk, that was presented in previous papers, we show that the constructions used to establish that correspondence lead to very general proofs of existence of solutions of the Dirichlet problem on the plane. At first, this establishes the existence of solutions for almost arbitrary integrable real functions on the unit circle, including functions which are discontinuous and unbounded. The proof of existence is then generalized to a large class of non-integrable real functions on the unit circle. Further, the proof of existence is generalized to real functions on a large class of other boundaries on the plane, by means of conformal transformations.

1 Introduction

In previous papers [1–4] we have shown that there is a correspondence between, on the one hand, integrable real functions, singular Schwartz distributions and non-integrable real functions which are locally integrable almost everywhere, and on the other hand, inner analytic functions within the open unit disk of the complex plane. This correspondence is based on the complex-analytic structure within the unit disk of the complex plane, which we introduced in [1]. In order to establish this correspondence for integrable and non-integrable real functions, we presented in [1] and [4] certain constructions which, given just such a real function, produce from it a unique corresponding inner analytic function.

In this paper we will show that these constructions have, as collateral consequences, the establishment of very general constructive proofs of the existence of the solution of the Dirichlet boundary value problem for the Laplace equation on regions of the plane. We will first establish the proof for integrable real functions on the unit circle, then generalize it to non-integrable real functions which are locally integrable almost everywhere on that circle. Furthermore, with the use of conformal transformations it is possible to generalize the proof to integrable and non-integrable real function on other boundaries on the plane. We will first establish this generalization for a large class of differentiable curves, and then, with a single weak additional limitation on the real functions, for a large class of curves that can be non-differentiable at a finite set of points, such as polygons.

For ease of reference, we include here a one-page synopsis of the complex-analytic structure introduced in [1]. It consists of certain elements within complex analysis [5], as well as of their main properties.

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Synopsis: The Complex-Analytic Structure

An *inner analytic function* $w(z)$ is simply a complex function which is analytic within the open unit disk. An inner analytic function that has the additional property that $w(0)=0$ is a *proper inner analytic function*. The *angular derivative* of an inner analytic function is defined by

$$w^{\bullet}(z) = \imath z \, \frac{dw(z)}{dz}. \tag{1}$$

By construction we have that $w^{\bullet}(0)=0$, for all $w(z)$. The *angular primitive* of an inner analytic function is defined by

$$w^{-1\bullet}(z) = -\imath \int_0^z dz' \, \frac{w(z')-w(0)}{z'}. \tag{2}$$

By construction we have that $w^{-1\bullet}(0)=0$, for all $w(z)$. In terms of a system of polar coordinates (ρ,θ) on the complex plane, these two analytic operations are equivalent to differentiation and integration with respect to θ, taken at constant ρ. These two operations stay within the space of inner analytic functions, they also stay within the space of proper inner analytic functions, and they are the inverses of one another. Using these operations, and starting from any proper inner analytic function $w^{0\bullet}(z)$, one constructs an infinite *integral-differential chain* of proper inner analytic functions,

$$\left\{ \ldots, w^{-3\bullet}(z), w^{-2\bullet}(z), w^{-1\bullet}(z), w^{0\bullet}(z), w^{1\bullet}(z), w^{2\bullet}(z), w^{3\bullet}(z), \ldots \right\}. \tag{3}$$

Two different such integral-differential chains cannot ever intersect each other. There is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, namely the null chain, in which all members are the null function $w(z) \equiv 0$.

A general scheme for the classification of all possible singularities of inner analytic functions is established. A singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity*. Angular integration takes soft singularities to other soft singularities, and angular differentiation takes hard singularities to other hard singularities.

Gradations of softness and hardness are then established. A hard singularity that becomes a soft one by means of a single angular integration is a *borderline hard* singularity, with degree of hardness zero. The *degree of softness* of a soft singularity is the number of angular differentiations that result in a borderline hard singularity, and the *degree of hardness* of a hard singularity is the number of angular integrations that result in a borderline hard singularity. Singularities which are either soft or borderline hard are integrable ones. Hard singularities which are not borderline hard are non-integrable ones.

Given an integrable real function $f(\theta)$ on the unit circle, one can construct from it a unique corresponding inner analytic function $w(z)$. Real functions are obtained through the $\rho \to 1_{(-)}$ limit of the real and imaginary parts of each such inner analytic function and, in particular, the real function $f(\theta)$ is obtained from the real part of $w(z)$ in this limit. The pair of real functions obtained from the real and imaginary parts of one and the same inner analytic function are said to be mutually Fourier-conjugate real functions.

Singularities of real functions can be classified in a way which is analogous to the corresponding complex classification. Integrable real functions are typically associated with inner analytic functions that have singularities which are either soft or at most borderline hard. This ends our synopsis.

The treatment of the Dirichlet problem is usually developed under the hypothesis that the boundary conditions are given by continuous real functions at the boundary, leading to solutions which are continuous and twice differentiable, with continuous derivatives, within the interior. For the two-dimensional problems we will consider here, we will be able to relax the conditions on the real functions at the boundary, accepting as valid boundary conditions real functions which may not be continuous, and not even bounded, at a finite set of boundary points. In order to allow for this, the condition that the solution within the interior reproduces the boundary condition everywhere at the boundary will have to be relaxed to the reproduction only almost everywhere at the boundary. On the other hand, it will also follow from the proofs offered that the solutions within the interior are not only continuous and twice differentiable, but in fact that they are always infinitely differentiable functions, on both their arguments.

We begin our work in this paper in Section 2, by establishing the existence theorem for boundary conditions given by integrable real functions on the unit circle. This is followed, in Section 3, by an extension of the existence theorem to non-integrable real functions on the unit circle, which are, however, locally integrable almost everywhere there. In Section 4 we discuss the conformal transformations that are required for the further versions of the existence theorem, that are established in the subsequent sections. In Section 5 we establish the existence theorem for integrable real functions on almost arbitrary differentiable simple closed curves on the plane. In Section 6 this existence theorem is extended to the case of integrable real functions on non-differentiable simple closed curves on the plane, curves which have, however, at most a finite set of points of non-differentiability. In Section 7 the existence theorem is further extended, this time to non-integrable real functions, as qualified above, on differentiable simple closed curves. Finally, in Section 8 we present the last and most general extension of the existence theorem, to non-integrable real functions, as qualified above, on non-differentiable simple closed curves, also as qualified above.

When we discuss real functions in this paper, some properties will be globally assumed for these functions, just as was done in the previous papers [1–4] leading to this one. These are rather weak conditions to be imposed on these functions, that will be in force throughout this paper. It is to be understood, without any need for further comment, that these conditions are valid whenever real functions appear in the arguments. These weak conditions certainly hold for any real functions that are obtained as restrictions of corresponding inner analytic functions to the unit circle, or to other simple closed curves with finite total length.

The most basic global condition we will impose is that the real functions must be measurable in the sense of Lebesgue, with the usual Lebesgue measure [6, 7]. The second global condition we will impose is that the real functions have no removable singularities. The third and last global condition is that the number of hard singularities of the real functions on their domains of definition be finite, and hence that they be all isolated from one another. There will be no limitation on the number of soft singularities.

The material contained in this paper is a development, reorganization and extension of some of the material found, sometimes still in rather rudimentary form, in the papers [8–12].

2 Integrable Real Functions on the Unit Circle

In a previous paper [1] we have shown that, given an integrable real function on the unit circle, one can define from it a unique inner analytic function whose real part reproduces that real function when restricted to the unit circle. What follows is an outline of the construction of this inner analytic function. Given the integrable real function $f(\theta)$, we define from it, by means of the usual integrals, the Fourier coefficients α_0, α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, and from those coefficients we define the complex Taylor coefficients $c_0 = \alpha_0/2$ and $c_k = \alpha_k - \imath\beta_k$, for $k \in \{1, 2, 3, \ldots, \infty\}$. As was shown in [1], the complex power series generated from these coefficients,

$$S(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{4}$$

always converges to an inner analytic function $w(z)$ within the open unit disk,

$$w(z) = u(\rho, \theta) + \imath v(\rho, \theta). \tag{5}$$

As was also shown in [1], the $\rho \to 1_{(-)}$ limit of the real part $u(\rho, \theta)$ reproduces $f(\theta)$ at all points on the unit circle where $w(z)$ does not have hard singularities. It does have hard singularities at all points where $f(\theta)$ does, so we are led to impose that these must be finite in number. However, in some special cases $w(z)$ may have hard singularities at points where $f(\theta)$ does not, and therefore we are led to assume independently that the number of hard singularities of $w(z)$ is finite. For all integrable real functions $f(\theta)$ that correspond to inner analytic functions $w(z)$ which have at most a finite number of hard singularities on the unit circle, we have that

$$f(\theta) = \lim_{\rho \to 1_{(-)}} u(\rho, \theta), \tag{6}$$

almost everywhere. Since, being the real part of an analytic function, the real function $u(\rho, \theta)$ is a harmonic function defined on the plane, and thus satisfies the Laplace equation within the open unit disk,

$$\nabla^2 u(\rho, \theta) = 0, \tag{7}$$

this construction establishes the existence of a solution of the Dirichlet problem on the unit disk or, more precisely, the existence of a solution of the Dirichlet boundary value problem of the Laplace equation on the unit disk. Given the boundary condition $u(1, \theta) = f(\theta)$, the solution is $u(\rho, \theta)$, which by construction satisfies the Laplace equation within the open unit disk and which, also by construction, assumes the values $f(\theta)$ on the unit circle, at least almost everywhere.

Note that, since $f(\theta)$ may have isolated singular points where it diverges to infinity, at which it is, therefore, not well defined, it is clear that $u(\rho, \theta)$ can reproduce $f(\theta)$ only almost everywhere. However, $u(\rho, \theta)$ may fail to reproduce $f(\theta)$ at points other than its hard singularities, namely points where $w(z)$ has hard singularities but $f(\theta)$ happens to have soft ones, due to the way in which the complex singularities of $w(z)$ are oriented with respect to the directions tangent to the unit circle at these singular points. In this case the $\rho \to 1_{(-)}$ limit of $u(\rho, \theta)$ does not exist at such points, and therefore at these points it is not possible to recover the values of $f(\theta)$ in this way.

Note also that, if one introduces some removable singularities of $f(\theta)$ at some points on the unit circle, then this does not change the Fourier coefficients α_0, α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, since these are given by integrals, which implies that it does not change the Taylor coefficients c_0 and c_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, and therefore that it also does not change the corresponding inner analytic function $w(z)$. It follows that $u(\rho, \theta)$ cannot reproduce $f(\theta)$ at these points, if arbitrary real values are attributed to $f(\theta)$ at them. Therefore, we are led to also assume that $f(\theta)$ has no such removable singularities or, equivalently, we are led to assume that all such removable singularities have been removed, and the function redefined by continuity at these trivial singular points.

Here is, then, a complete and precise statement of the Dirichlet problem on the unit disk, followed by the complete set of assumptions to be imposed on $f(\theta)$ in order to ensure the existence of the solution of that problem.

Definition 1: *The Dirichlet Problem on the Unit Disk*

Given the unit circle on the complex plane and a real function $f(\theta)$ defined on it, the existence problem of the Dirichlet boundary value problem of the Laplace equation on the unit disk is to show that a function $u(\rho, \theta)$ exists such that it satisfies

$$\nabla^2 u(\rho, \theta) = 0, \tag{8}$$

within the open unit disk, and such that it also satisfies

$$u(1, \theta) = f(\theta), \tag{9}$$

almost everywhere on the unit circle.

In this section, using our results from previous papers, we will establish the following theorem.

Theorem 1: *Given a real function $f(\theta)$ at the boundary of the unit disk, that satisfies the list of conditions described below, there is a solution $u(\rho, \theta)$ of the Dirichlet problem of the Laplace equation within the open unit disk, that assumes the values $f(\theta)$ almost everywhere at its boundary, the unit circle.*

Proof 1.1:

According to the construction introduced in [1] and reviewed above, which provides $u(\rho, \theta)$ starting from $f(\theta)$, the function $u(\rho, \theta)$ that results from that construction is a solution to this problem so long as $f(\theta)$ satisfies the following set of conditions, which ensure that the construction of the inner analytic function $w(z)$ from the real function $f(\theta)$ succeeds, and that the real part $u(\rho, \theta)$ of $w(z)$ reproduces $f(\theta)$ almost everywhere oven the unit circle in the $\rho \to 1_{(-)}$ limit. Apart from the global conditions that the real function $f(\theta)$ be a Lebesgue-measurable function and that it have no removable singularities, the conditions on $f(\theta)$ for this theorem are as follows.

1. The real function $f(\theta)$ is integrable on the unit circle.

2. The number of hard singularities of the corresponding inner analytic function $w(z)$ is finite.

This completes the proof of Theorem 1.

Note that the last condition implies, in particular, that the number of hard singularities of $f(\theta)$, where it is either discontinuous or diverges to infinity, is also finite. Note also that, since the function must be integrable, any hard singularities where it diverges to infinity must be integrable ones, in the real sense of the terms involved. This requires that these hard singularities be all isolated from each other, so that there is a neighborhood around each one of them within which the two lateral asymptotic limits of integrals can be considered. It is important to emphasize that the conditions above over the real functions $f(\theta)$ include functions which are non-differentiable at any number of points, discontinuous at a finite number of points, and unbounded at a finite number of points, thus constituting a rather large set of boundary conditions.

3 Non-Integrable Real Functions on the Unit Circle

In a previous paper [4] we showed that the correspondence between real functions and inner analytic functions established in [1] can be extended to non-integrable real functions, so long as these functions are locally integrable almost everywhere, and so long as the non-integrable hard singularities of the corresponding inner analytic functions have finite degrees of hardness. The definition of local integrability almost everywhere on the unit circle is as follows.

Definition 2: *A real function $f(\theta)$ is locally integrable almost everywhere on the unit circle if it is integrable on every closed interval $[\theta_\ominus, \theta_\oplus]$ contained within that domain, that does not contain any of the points where the function has non-integrable hard singularities, of which there is a finite number.*

Although the construction used in this case, which is given in [4], is considerably more involved than the one for the case of integrable real functions, it is still true that given such a non-integrable real function one can define a unique inner analytic function $w(z)$ that corresponds to it, as well as a unique and complete set of complex Taylor coefficients $c_0 = \alpha_0/2$ and $c_k = \alpha_k - \imath\beta_k$, for $k \in \{1, 2, 3, \ldots, \infty\}$, and thus a corresponding unique and complete set of Fourier coefficients α_0, α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, that are associated to it, despite the fact that the real function is not integrable. From the real part of this inner analytic function one can, once again, recover the real function almost everywhere by taking the $\rho \to 1_{(-)}$ limit to the unit circle. Therefore we have at hand all that we need in order to implement the proof of existence in this more general case.

In this section, using again our results from previous papers, we will establish the following theorem.

Theorem 2: *Given a real function $f(\theta)$ at the boundary of the unit disk, that satisfies the list of conditions described below, there is a solution $u(\rho, \theta)$ of the Dirichlet problem of the Laplace equation within the open unit disk, that assumes the values $f(\theta)$ almost everywhere at its boundary, the unit circle.*

Proof 2.1:

The argument is the same as the one used before for Theorem 1 in Section 2, in the case of integrable real functions, but using now the construction presented in [4], instead of the one presented in [1]. Due to this, the only change with respect to that previous case is that our list of conditions on the real functions can now be upgraded to the following, still including the previous case.

1. The real function $f(\theta)$ is locally integrable almost everywhere on the unit circle, including the cases in which this function is globally integrable there.

2. The number of hard singularities of the corresponding inner analytic function $w(z)$ is finite.

3. The hard singularities of the corresponding inner analytic function $w(z)$ have finite degrees of hardness.

This completes the proof of Theorem 2.

In this way we have generalized the proof of existence of the Dirichlet problem from boundary conditions given by integrable real functions to others given by a certain class of non-integrable real functions. Note that the degree of hardness in the previous case, that of borderline hard singularities of integrable real functions, is simply zero. In other words, if all the hard singularities are borderline hard ones, then the function is simply integrable. Therefore, this theorem is a strict generalization of the previous one, and contains it.

4 Conformal Transformations to Other Curves

As we will see in the subsequent sections, it is possible to extend the proof of existence of the Dirichlet problem to boundaries other than the unit circle, through the use of conformal transformations. Therefore, as a preliminary to the proof of further versions of the existence theorem, in this section we will describe such conformal transformations and examine some of their well-known properties, targeting their use here. In order to do this, consider two complex variables z_a and z_b and the corresponding complex planes, a complex analytic function $\gamma(z)$ defined on the complex plane z_a with values on the complex plane z_b, and its inverse function, which is a complex analytic function $\gamma^{(-1)}(z)$ defined on the complex plane z_b with values on the complex plane z_a,

$$\begin{aligned} z_b &= \gamma(z_a), \\ z_a &= \gamma^{(-1)}(z_b). \end{aligned} \tag{10}$$

Consider a bounded and simply connected open region S_a on the complex plane z_a and its image S_b under $\gamma(z)$, which is a similar region on the complex plane z_b. It can be shown that if $\gamma(z)$ is analytic on S_a, is invertible there, and its derivative has no zeros there, then its inverse function $\gamma^{(-1)}(z)$ has these same three properties on S_b, and the mapping between the two complex planes established by $\gamma(z)$ and $\gamma^{(-1)}(z)$ is conformal, in the sense that it preserves the angles between oriented curves at points where they cross each other. Note that this mapping is a bijection between the two regions, and establishes an equivalence relation that can be extended in a transitive way to other regions.

Consider now that the regions under consideration are the interiors of simple closed curves. One of these curves will be the unit circle C_a on the complex plane z_a, and the other will be a given curve C_b on the complex plane z_b. We will assume that the curve C_b has finite total length, for two reasons, one being to ensure that the interior of the curve is a bounded set, and the other being to ensure that the integrals of real functions over the curve C_b are integrals over a finite-length, compact domain. Since $\gamma(z_a)$, being analytic, is in particular a continuous function, the image on the z_b plane of the unit circle C_a on the z_a plane must be a continuous closed curve C_b. We can also see that C_b must be a simple curve, because the fact that $\gamma(z_a)$ is invertible on C_a means that it cannot have the same value at two different points of C_a, and therefore no two points of C_b can be the same. Consequently, the curve C_b cannot self-intersect.

We thus see that, so far, we are restricted to simple closed curves C_b with finite total lengths. However, there are further limitations on the curves, implied by our hypotheses. Since the transformation is conformal and thus preserves angles, it follows that in this case the smooth unit circle C_a is mapped onto another equally differentiable circuit C_b. One can see this by considering the angles between tangents to the curve C_b at pairs of neighboring points, the corresponding elements on the curve C_a, and the limit of these angles when the two points tend to each other, given that the transformation is conformal. Therefore, with such limitations one cannot map the unit circle onto a square or any other polygon. This limitation can be lifted by allowing the derivative of $\gamma(z_a)$ to have a finite number of isolated zeros on the curve C_a, which then implies that the derivative of $\gamma^{(-1)}(z_b)$ will have a finite number of corresponding isolated singular points on C_b.

Let us assume that the unit circle C_a is described by the real arc-length parameter θ on the z_a plane, and that the curve C_b is described by a corresponding real parameter λ on the z_b plane. Let us assume also that λ is chosen in such a way that $|d\lambda| = |dz_b|$ over the curve C_b, just as $|d\theta| = |dz_a|$ over C_a, which means that λ is also an arc-length parameter. Since every point z_a on the curve C_a is mapped by the conformal transformation onto a corresponding point z_b on the curve C_b, and since a point z_a on C_a is described by a certain value of θ, while the corresponding point z_b on C_b is described by a certain value of λ, it is clear that the complex conformal transformation induces a corresponding real transformation between the values of θ and the values of λ,

$$\begin{aligned} z_b &= \gamma(z_a) \quad \Rightarrow \\ \lambda &= g(\theta), \end{aligned} \tag{11}$$

where the real function $g(\theta)$ is continuous, differentiable and invertible on C_a. We will refer to the function $g(\theta)$ as the real transformation induced on the curve C_a by the complex conformal transformation $\gamma(z_a)$. The same is true for the inverse transformation, which induces the inverse function of $g(\theta)$, on the curve C_b,

$$\begin{aligned} z_a &= \gamma^{(-1)}(z_b) \quad \Rightarrow \\ \theta &= g^{(-1)}(\lambda), \end{aligned} \tag{12}$$

where the real function $g^{(-1)}(\lambda)$ is continuous and invertible on C_b, and also differentiable so long as C_b is a differentiable curve. Before we proceed, we must now consider in more detail the question of what is the set of curves C_b for which the structure described above can be set up. We assume that this curve is a simple closed curve of finite total length, and the question is whether or not this structure can be set up for an arbitrary such curve. Given the curve C_b, the only additional objects we need in order to do this is the conformal mapping $\gamma(z_a)$ and its inverse $\gamma^{(-1)}(z_b)$, between that curve and the unit circle C_a.

The existence of these transformation functions can be ensured as a consequence of the famous Riemann mapping theorem, and of the associated results relating to conformal mappings between regions of the complex plane. According to that theorem, a conformal transformation such as the one we described here exists between any bounded simply connected open set of the plane and the open unit disk. In addition to this, one can show that this conformal mapping can be extended to the respective boundaries as a continuous function so long as the boundary curve C_b satisfies a certain condition [13].

The condition on C_b that implies the existence of the continuous extension to the boundary is that every point on that curve be what is called in the relevant literature a *simple point*. This means that no point of C_b can be a *multiple point*, which in essence is a point on the boundary that is accessible from the interior via two or more independent continuous paths contained in the interior, that cannot be continuously deformed into each other without crossing the boundary. We can see, therefore, that this condition has a topological character. Note that the presence of a multiple point on the boundary means that, even if the open set under consideration is simply connected, its *closure* will *not* be. Therefore, one way to formulate this condition is to simply state that the closure of the bounded simply connected open set must also be simply connected.

Since the existence of a multiple point at the boundary means that this boundary is not a simple curve, it follows that, under the limitations over C_b that we have here, the conformal mapping on the open unit disk can always be continuously extended to the unit circle, and hence from the interior of the curve C_b to that curve, which is mapped from the unit circle. Therefore, we conclude that it is a known fact that such a conformal transformation exists for all possible simple closed curves C_b with finite total lengths, and in particular for all such curves which are also differentiable, in which case the extension is also differentiable on the unit circle C_a. It is therefore not necessary to impose explicitly any additional hypotheses about the existence of the conformal transformation, regardless of whether or not the curves C_b under consideration are differentiable.

Let us close this section with a discussion of the nature of the singularities that appear in the case of simple closed curves C_b which are not differentiable at a finite set of points $z_{b,i}$, for $i \in \{1, \ldots, N\}$. The additional difficulty that appears in this case stems from the fact that, if the curve C_b is not differentiable at the points $z_{b,i}$, then the derivative of the transformation $\gamma(z_a)$ has isolated zeros at the corresponding points $z_{a,i}$ on the curve C_a, and therefore the derivative of the inverse transformation $\gamma^{(-1)}(z_b)$ has isolated hard singularities at the points $z_{b,i}$. In order to see how this comes about we start by noting that, since we have that

$$\gamma^{(-1)}(\gamma(z_a)) = z_a, \tag{13}$$

for all z_a on the closed unit disk, differentiating this equation we get, due to the chain rule,

$$\frac{d\gamma^{(-1)}}{dz_b}\frac{d\gamma}{dz_a} = 1. \tag{14}$$

Therefore, to every point $z_{a,i}$ on C_a where the derivative of the transformation has a zero corresponds a point $z_{b,i}$ on C_b where the derivative of the inverse transformation diverges to infinity. Taking absolute values we have, in terms of the arc-length parameters θ and λ,

$$\begin{aligned} \left|\frac{d\gamma^{(-1)}}{dz_b}\right|\left|\frac{d\gamma}{dz_a}\right| &= \left|\frac{dz_a}{dz_b}\right|\left|\frac{dz_b}{dz_a}\right| \\ &= \left|\frac{d\theta}{d\lambda}\right|\left|\frac{d\lambda}{d\theta}\right|, \end{aligned} \tag{15}$$

which implies that

$$\left|\frac{d\theta}{d\lambda}\right|\left|\frac{d\lambda}{d\theta}\right| = 1. \tag{16}$$

In fact, the real function in the left-hand side of this last equation has a removable singularity at every point where the first derivative in the product diverges and the second one is zero. Consequently, they can be removed by simply redefining the product by continuity at these points. When we approach one of the points $z_{a,i}$, which are characterized by the values θ_i of the parameter θ, along the curve C_a, we have that

$$\begin{aligned} \theta &\rightarrow \theta_i, \\ \lambda &\rightarrow \lambda_i \\ &\Downarrow \end{aligned}$$

$$
\begin{aligned}
\left|\frac{d\lambda}{d\theta}\right| &\rightarrow 0, \\
\left|\frac{d\theta}{d\lambda}\right| &\rightarrow \infty,
\end{aligned}
\tag{17}
$$

where the corresponding points $z_{b,i}$ are characterized by the values λ_i of the parameter λ, along the curve C_b. Although the derivative $d\theta/d\lambda$ does, therefore, have hard singularities at $z_{b,i}$, we can show that these are still integrable singularities. We simply integrate the expressions in either side of Equation (16) absolutely over C_a, thus obtaining

$$
\oint_{C_a} |d\theta| \left|\frac{d\theta}{d\lambda}\right| \left|\frac{d\lambda}{d\theta}\right| = 2\pi. \tag{18}
$$

If we now change variables in this integral from θ to λ, we get the integral over C_b

$$
\oint_{C_b} |d\lambda| \left|\frac{d\theta}{d\lambda}\right| = 2\pi. \tag{19}
$$

This shows that the real function appearing as the integrand in this integral is an integrable real function on C_b. Therefore the hard singularities where the derivative $d\theta/d\lambda$ diverges to infinity are integrable hard singularities, which therefore have degree of hardness zero. These are also referred to as borderline hard singularities. Note that, as a consequence, the corresponding singularities of the inverse real transformation $g^{(-1)}(\lambda)$ itself must be soft ones, with degrees of softness equal to one.

5 Integrable Real Functions on Differentiable Curves

In this section we will show how one can generalize the proof of existence of the solution of the Dirichlet problem on the unit disk, given by Theorem 1 in Section 2, to the case in which we have, as the boundary condition, integrable real functions defined on boundaries given by differentiable curves on the plane. In order to do this, the first thing we must do here is to establish the precise definition of the Dirichlet problem in this case.

Definition 3: *The Dirichlet Problem on a Given Curve and its Interior*

Given a simple closed curve C on the complex plane, described by a real arc-length parameter λ, and a real function $f(\lambda)$ defined on it, the existence problem of the Dirichlet boundary value problem of the Laplace equation on this curve and its interior is to show that a function $u(x,y)$ exists such that it satisfies

$$
\nabla^2 u(x,y) = 0, \tag{20}
$$

within the interior of C, and such that it also satisfies

$$
u(x,y) = f(\lambda), \tag{21}
$$

for $z = x + \imath y$ on C, thus corresponding to λ, almost everywhere over that curve.

Therefore, using the notation established in Section 4, let C_b be a differentiable simple closed curve on the complex z_b plane, with finite total length, and let us assume that a Dirichlet boundary value problem for the Laplace equation is given on the region whose boundary is the curve C_b, that is, let there be given also an integrable real function $f_b(\lambda)$ on C_b, that is, a function such that the integral

$$\oint_{C_b} d\lambda\, f_b(\lambda) \tag{22}$$

exists and is finite. The problem is then to establish the existence of a function $u_b(x, y)$ that satisfies $\nabla^2 u_b(x, y) = 0$ in the interior of the curve C_b and that assumes the values $f_b(\lambda)$ almost everywhere over that curve. In order to do this using the conformal transformation $\gamma(z)$ from the complex plane z_a to the complex plane z_b, we start by constructing a corresponding Dirichlet problem on the unit disk of the z_a plane, using the mapping between z_b and z_a provided by the conformal transformation $\gamma(z_a)$ and its inverse $\gamma^{(-1)}(z_b)$. We define a corresponding real function $f_a(\theta)$ on the unit circle C_a by simply transferring the values of $f_b(\lambda)$ through the use of the conformal mapping from point to point,

$$\begin{aligned} f_a(\theta) &= f_b(\lambda) \\ &= f_b(g(\theta)), \end{aligned} \tag{23}$$

where θ describes a point on C_a given by the complex number z_a, λ describes the corresponding point on C_b given by the complex number $z_b = \gamma(z_a)$, and $g(\theta)$ is the induced real transformation, so that we have that $\lambda = g(\theta)$ and that $\theta = g^{(-1)}(\lambda)$. We will start by establishing the following preliminary fact about the real function $f_a(\theta)$ defined as above from an integrable real function $f_b(\lambda)$.

Lemma 1: *Given a real function $f_b(\lambda)$ which is integrable on the differentiable simple closed curve C_b of finite total length, it follows that the corresponding function $f_a(\theta)$ defined on the unit circle C_a by $f_a(\theta) = f_b(\lambda)$ is integrable on that circle.*

Given that $f_b(\lambda)$ is integrable on C_b, we must show that $f_a(\theta)$ defined by the composition of $f_b(\lambda)$ with $g(\theta)$, is also integrable, that is, it is integrable on C_a. Since all real functions under discussion here are assumed to be Lebesgue-measurable, and since for such measurable functions defined on compact domains integrability and absolute integrability are equivalent conditions [6, 7], given that $f_b(\lambda)$ is integrable on C_b we have that the integral

$$\oint_{C_b} |d\lambda|\, |f_b(\lambda)| \tag{24}$$

exists and is finite. We must now show that the integral

$$\oint_{C_a} |d\theta|\, |f_a(\theta)| \tag{25}$$

exists and is finite, which is equivalent to the statement that $f_a(\theta)$ is integrable on C_a. Changing variables on this integral from θ to λ, and using the fact that by definition we have that $f_a(\theta) = f_b(\lambda)$, we obtain

$$\oint_{C_a} |d\theta|\, |f_a(\theta)| = \oint_{C_b} |d\lambda| \left|\frac{d\theta}{d\lambda}\right| |f_b(\lambda)|. \tag{26}$$

Since $f_b(\lambda)$ is integrable on C_b, and since the absolute value of the derivative shown exists and is finite, given that we have

$$\begin{aligned} \left|\frac{d\theta}{d\lambda}\right| &= \left|\frac{dz_a}{dz_b}\right| \\ &= \left|\frac{d\gamma^{(-1)}(z_b)}{dz_b}\right|, \end{aligned} \tag{27}$$

where $\gamma^{(-1)}(z_b)$ is analytic on C_b and therefore differentiable there, it follows that the absolute value of the derivative which appears in the integrand on the right-hand side of Equation (26) is a limited real function on C_b. Since $f_b(\lambda)$ is integrable on C_b, from this it follows that the whole integrand of the integral on the right-hand side of Equation (26), which is the product of a limited real function with an integrable real function, is itself an integrable real function on C_b, so that we may conclude that the integral in Equation (25) exists and is finite, and therefore that $f_a(\theta)$ is an integrable real function on C_a. This establishes Lemma 1.

In this section, using the results from the previous sections, we will establish the following theorem.

Theorem 3: *Given a differentiable simple closed curve C of finite total length on the complex plane $z = x + \imath y$, given the invertible conformal transformation $\gamma(z)$ whose derivative has no zeros on the closed unit disk, that maps it from the unit circle, and given a real function $f(\lambda)$ on that curve, that satisfies the list of conditions described below, there is a solution $u(x, y)$ of the Dirichlet problem of the Laplace equation within the interior of that curve, that assumes the given values $f(\lambda)$ almost everywhere on the curve.*

Proof 3.1:

The proof consists of using the conformal transformation between the closed unit disk and the union of the curve C with its interior, a transformation which according to the analysis in Section 4 always exists, to map the given boundary condition on C onto a corresponding boundary condition on the unit circle, then using the proof of existence established before by Theorem 1 in Section 2 for the closed unit disk to establish the existence of the solution of the corresponding Dirichlet problem on that disk, and finally using once more the conformal transformation to map the resulting solution back to C and its interior, showing in the process that one obtains in this way the solution of the Dirichlet problem there. The list of conditions on the real functions is now the following.

1. The real function $f(\lambda)$ is integrable on C.
2. The number of hard singularities on the unit circle of the corresponding inner analytic function $w(z)$ on the unit disk is finite.

According to the preliminary result established in Lemma 1, if the real function $f_b(\lambda)$ satisfies these conditions on C_b, then $f_a(\theta)$ is an integrable real function on C_a. Therefore, due to the existence theorem of the Dirichlet problem on the unit disk of the plane z_a, which was established by Theorem 1 in Section 2, we know that there is an inner analytic function $w_a(z_a)$ such that its real part $u_a(\rho, \theta)$ is harmonic within the open unit disk and also satisfies $u_a(1, \theta) = f_a(\theta)$ almost everywhere at the boundary C_a. Now, by composing $w_a(z_a)$ with the inverse conformal transformation $\gamma^{(-1)}(z_b)$ we get on the z_b plane the complex function

$$\begin{aligned} w_b(z_b) &= w_a(z_a) \\ &= w_a\left(\gamma^{(-1)}(z_b)\right), \end{aligned} \tag{28}$$

which corresponds to simply transferring back the values of $w_a(z_a)$, by the use of the conformal mapping from point to point, while we also have, of course, the corresponding inverse real transformation at the boundary,

$$
\begin{aligned}
f_b(\lambda) &= f_a(\theta) \\
&= f_a\left(g^{(-1)}(\lambda)\right).
\end{aligned} \tag{29}
$$

Given that the composition of two analytic functions is also analytic, in their chained domain of analyticity, and since $\gamma^{(-1)}(z_b)$ is analytic in the interior of the curve C_b, and also since $w_a(z_a)$ is analytic in the interior of the curve C_a, we conclude that $w_b(z_b)$ is analytic in the interior of the curve C_b. Therefore, the real part $u_b(x,y)$ of $w_b(z_b)$ is harmonic and thus satisfies

$$
\nabla^2 u_b(x,y) = 0 \tag{30}
$$

in the interior of C_b, while by construction the fact that we have $u_a(\rho,\theta) = f_a(\theta)$ for the points given by $z_a = \rho\exp(\imath\theta)$ almost everywhere on C_a, where $\rho = 1$, implies that we also have

$$
u_b(x,y) = f_b(\lambda), \tag{31}
$$

for the corresponding points given by $z_b = x + \imath y$ almost everywhere on C_b, and which thus correspond to λ. This establishes the existence, by construction, of the solution of the Dirichlet problem on the z_b plane, under our current hypotheses. This completes the proof of Theorem 3.

In this way we have generalized the proof of existence of the Dirichlet problem from the unit circle to all differentiable simple closed curves with finite total lengths on the plane, for boundary conditions given by integrable real functions.

6 Integrable Real Functions on Non-Differentiable Curves

In this section we will extend the existence theorem of the Dirichlet problem on the unit disk, given by Theorem 1 in Section 2, to regions bounded by simple closed curves C_b which are not differentiable at a finite set of points $z_{b,i}$, for $i \in \{1,\ldots,N\}$. We will still use the conformal transformation known to exist between the open unit disk and the interior of any such curve, as well as its continuous extension to the respective boundaries, where the extension is also differentiable almost everywhere, with the exception of a finite set of singularities of the inverse conformal transformation, at the points $z_{b,i}$, where the inverse conformal transformation still exists but is not differentiable. Note that in Section 4 we established the existence of the conformal mapping $\gamma(z)$ for all simple closed curves C_b with finite total lengths, regardless of whether or not the curves are differentiable.

The additional difficulty that appears in this case stems from the fact that, if the curve C_b is not differentiable at the points $z_{b,i}$, then the derivative of the transformation $\gamma(z_a)$ has isolated zeros at the corresponding points $z_{a,i}$ on the curve C_a, and therefore the derivative of the inverse transformation $\gamma^{(-1)}(z_b)$ has isolated hard singularities at the points $z_{b,i}$, as was discussed in Section 4. This will require that we impose one additional limitation on the real functions giving the boundary conditions, namely that any integrable hard singularities where they diverge to infinity do not coincide with any of the points $z_{b,i}$.

The precise definition of the Dirichlet problem in this case is the same one given in Definition 3, in Section 5. We will start by establishing the following preliminary fact about the real function $f_a(\theta)$ defined from an integrable real function $f_b(\lambda)$.

Lemma 2: *Given a real function $f_b(\lambda)$ which is integrable on the simple closed curve C_b of finite total length, which is not differentiable at a finite set of points $z_{b,i}$, for $i \in \{1, \ldots, N\}$, and given that the integrable hard singularities of $f(\lambda)$ where it diverges to infinity are not located at any of the points $z_{b,i}$ where the curve is non-differentiable, it follows that the corresponding function $f_a(\theta)$ defined on the unit circle C_a by $f_a(\theta) = f_b(\lambda)$ is integrable on that circle.*

Given that $f_b(\lambda)$ is integrable on C_b, we must decide whether or not $f_a(\theta)$ is also integrable, that is, whether it is integrable on C_a. Once again, given that $f_b(\lambda)$ is integrable on C_b we have that the integral

$$\oint_{C_b} |d\lambda|\, |f_b(\lambda)| \tag{32}$$

exists and is finite. We must now determine whether or not the integral

$$\oint_{C_a} |d\theta|\, |f_a(\theta)| \tag{33}$$

exists and is finite, which is equivalent to the statement that $f_a(\theta)$ is integrable on C_a. Changing variables on this integral from θ to λ we obtain once again

$$\oint_{C_a} |d\theta|\, |f_a(\theta)| = \oint_{C_b} |d\lambda| \left|\frac{d\theta}{d\lambda}\right| |f_b(\lambda)|. \tag{34}$$

Since both the absolute value of the derivative shown and the function $f_b(\lambda)$ are integrable on C_b, and since the integrable borderline hard singular points where either one of these two real functions diverges to infinity do not coincide, we have that the integrand of the integral on the right-hand side of this equation is also an integrable real function, and thus that the integral exists and is finite. We can see that the integrand is an integrable real function because around each integrable hard singular point of either one of the two real functions involved there is a neighborhood where the other real function is limited. Since the product of a limited real function with an integrable real function is also an integrable real function, we may conclude that the integrand is locally integrable *everywhere* on C_b, and therefore globally integrable there, so that the integral above exists and is finite. It thus follows that $f_a(\theta)$ is an integrable real function on C_a. This establishes Lemma 2.

In this section, using again the results from the previous sections, we will establish the following theorem.

Theorem 4: *Given a simple closed curve C of finite total length on the complex plane $z = x + \imath y$, which is non-differentiable at a given finite set of points z_i, for $i \in \{1, \ldots, N\}$, given the conformal transformation $\gamma(z)$ that maps it from the unit circle, whose derivative has zeros on the unit circle at the corresponding set of points, and given a real function $f(\lambda)$ on that curve, that satisfies the list of conditions described below, there is a solution $u(x, y)$ of the Dirichlet problem of the Laplace equation within the interior of that curve, that assumes the given values $f(\lambda)$ almost everywhere on the curve.*

Proof 4.1:

Just as in the previous case, in Section 5, the proof consists of using the conformal transformation between the closed unit disk and the union of the curve C with its interior, which according to the analysis in Section 4 always exists, to map the given boundary condition on C onto a corresponding boundary condition on the unit circle, then using the proof of existence established before by Theorem 1 in Section 2 for the closed unit disk to establish the existence of the solution of the corresponding Dirichlet problem on that disk, and finally using once more the conformal transformation to map the resulting solution back to C and its interior, thus obtaining the solution of the original Dirichlet problem. The list of conditions on the real functions is now the following.

1. The real function $f(\lambda)$ is integrable on C.
2. The number of hard singularities on the unit circle of the corresponding inner analytic function $w(z)$ on the unit disk is finite.
3. The integrable hard singularities of $f(\lambda)$ where it diverges to infinity are not located at any of the points where the curve C is non-differentiable.

The rest of the proof is identical to that of the previous case, in Section 5. Therefore, once again we may conclude that, due to the existence theorem of the Dirichlet problem on the unit disk of the plane z_a, which was established by Theorem 1 in Section 2, we know that there is an inner analytic function $w_a(z_a)$ such that its real part $u_a(\rho, \theta)$ is harmonic within the open unit disk and satisfies $u_a(1, \theta) = f_a(\theta)$ almost everywhere at the boundary C_a. Just as in Section 5, we get on the z_b plane the complex function $w_b(z_b)$ which is analytic in the interior of the curve C_b. Therefore, the real part $u_b(x, y)$ of $w_b(z_b)$ is harmonic and thus satisfies

$$\nabla^2 u_b(x, y) = 0 \tag{35}$$

in the interior of C_b, while we also have that

$$u_b(x, y) = f_b(\lambda), \tag{36}$$

almost everywhere on C_b. This establishes the existence, by construction, of the solution of the Dirichlet problem on the z_b plane, under our current hypotheses. This completes the proof of Theorem 4.

In this way we have generalized the proof of existence of the Dirichlet problem from the unit circle to all simple closed curves with finite total lengths on the plane, that can be either differentiable or non-differentiable on at most a finite set of points, still for boundary conditions given by integrable real functions.

7 Non-Integrable Real Functions on Differentiable Curves

In this section we will show how one can generalize the proof of existence of the Dirichlet problem on the unit disk, given by Theorem 2 in Section 3, to the case in which we have, as the boundary condition, non-integrable real functions $f_b(\lambda)$ defined on boundaries given by differentiable curves C_b on the plane. We will be able to do this if the non-integrable real functions, despite being non-integrable over the whole curves C_b, are however locally integrable almost everywhere on those curves, and if, in addition to this, the non-integrable hard singularities of the inner analytic functions involved have finite degrees of hardness. The definition of the concept of local integrability almost everywhere is similar to that given for the unit circle by Definition 2, in Section 3. In our case here the precise definition of local integrability almost everywhere is as follows.

Definition 4: *A real function $f(\lambda)$ is locally integrable almost everywhere on the curve C described by the arc-length parameter λ if it is integrable on every closed interval $[\lambda_\ominus, \lambda_\oplus]$ contained within that domain, that does not contain any of the points where the function has non-integrable hard singularities, of which there is a finite number.*

The proof will follow the general lines of the one given for integrable real functions in Section 5, with the difference that, since the real functions $f_b(\lambda)$ are assumed to be non-integrable on C_b, but locally integrable almost everywhere there, instead of showing that the corresponding functions $f_a(\theta)$ on the unit circle C_a are integrable there, we will show that they are locally integrable almost everywhere there. In addition to this, instead of using the result for integrable real function on the unit circle, which was given by Theorem 1 in Section 2, we will use the corresponding result for non-integrable real functions which are locally integrable almost everywhere on the unit circle, which was given by Theorem 2 in Section 3. Since that result depends also on the non-integrable hard singularities of the real functions having finite degrees of hardness, we will also show that the hypothesis that the functions $f_b(\lambda)$ have this property implies that the corresponding functions $f_a(\theta)$ have the same property as well. In order to do this we will use the technique of *piecewise integration* which was introduced and employed in [4], where it played a crucial role.

We will start by showing the following preliminary fact about a real function $f_a(\theta)$ defined from a real function $f_b(\lambda)$ which is locally integrable almost everywhere on C_b, and which has non-integrable hard singularities at the finite set of points $z_{b,j}$, for $j \in \{1, \ldots, M\}$.

Lemma 3: *Given a real function $f_b(\lambda)$ which is integrable on a given closed interval I_b on C_b, it follows that the corresponding function $f_a(\theta)$ defined on the unit circle C_a by $f_a(\theta) = f_b(\lambda)$ is integrable on the corresponding closed interval I_a on C_a, which is mapped from I_b by the inverse conformal transformation $\gamma^{(-1)}(z_b)$.*

Since $f_b(\lambda)$ is integrable on I_b we have that

$$\int_{I_b} |d\lambda|\, |f_b(\lambda)| \tag{37}$$

exists and is finite. If we now consider the integral

$$\int_{I_a} |d\theta|\, |f_a(\theta)|, \tag{38}$$

and transform variables from θ to λ, recalling that $f_a(\theta) = f_b(\lambda)$, we get

$$\int_{I_a} |d\theta|\, |f_a(\theta)| = \int_{I_b} |d\lambda| \left| \frac{d\theta}{d\lambda} \right| |f_b(\lambda)|. \tag{39}$$

The absolute value of the derivative shown exists and is finite on I_b, given that

$$\left| \frac{d\theta}{d\lambda} \right| = \left| \frac{d\gamma^{(-1)}(z_b)}{dz_b} \right|, \tag{40}$$

where $\gamma^{(-1)}(z_b)$ is analytic on C_b and therefore differentiable there. We also have that $f_b(\lambda)$ is integrable on I_b. It follows that, since the integrand in the right-hand side of Equation (39) is the product of a limited real function with an integrable real function, and therefore is itself an integrable real function, the integral in Equation (39) exists and is finite, thus implying that $f_a(\theta)$ is integrable on the closed interval I_a. This establishes Lemma 3.

As an immediate consequence of this preliminary result, under the conditions that we have here, the hypothesis that $f_b(\lambda)$ is locally integrable almost everywhere on C_b, with the exclusion of the finite set of points $z_{b,j}$, implies that $f_a(\theta)$ is locally integrable almost everywhere on C_a, with the exclusion of the corresponding finite set of points $z_{a,j}$.

We must now discuss the issue of the degrees of hardness of the non-integrable hard singularities of the function $f_a(\theta)$ on C_a. Since by hypothesis $f_b(\lambda)$ has non-integrable hard singularities at the points $z_{b,j}$, it clearly follows that $f_a(\theta)$ also has hard singularities at the corresponding points $z_{a,j}$, which may be non-integrable ones. In order to discuss their degrees of hardness we will use the technique of piecewise integration, that is, we will consider sectional integrals of $f_a(\theta)$ on closed intervals contained within a neighborhood of the point $z_{a,j}$ where it has a single isolated hard singularity. Let us show the following preliminary fact about a real function $f_a(\theta)$ defined from a real function $f_b(\lambda)$ which is locally integrable almost everywhere on C_b, and which has non-integrable hard singularities with finite degrees of hardness at the finite set of points $z_{b,j}$.

Lemma 4: *Given a real function $f_b(\lambda)$ which has an isolated non-integrable hard singularity with finite degree of hardness at a point $z_{b,j}$ on C_b, it follows that the corresponding function $f_a(\theta)$ defined on the unit circle C_a by $f_a(\theta) = f_b(\lambda)$ has an isolated non-integrable hard singularity with finite degree of hardness at the corresponding point $z_{a,j}$ on C_a.*

Since the real functions must diverge to infinity at non-integrable hard singular points, the fact that $f_a(\theta)$ has an isolated hard singularity on $z_{a,j}$ is immediate. Since these singularities are all isolated from each other, there is on C_b a neighborhood of the point $z_{b,j}$ within which there are no other non-integrable hard singularities of $f_b(\lambda)$. Since the conformal mapping is continuous, it follows that there is on C_a a neighborhood of the corresponding point $z_{a,j}$ within which there are no other hard singularities of $f_a(\theta)$. Given that the point $z_{a,j}$ corresponds to the angle θ_j, let the closed interval $[\theta_{\ominus,j}, \theta_{\oplus,j}]$ contain the point $z_{a,j}$ and be contained in this neighborhood, so that we have

$$\theta_{\ominus,j} < \theta_j < \theta_{\oplus,j}, \tag{41}$$

where the sole hard singularity of $f_a(\theta)$ which is contained within this interval is the one at the point θ_j. Let us now consider a pair of closed intervals contained within this neighborhood, one to the left and one to the right of the point θ_j, so that we have

$$\begin{aligned} I_{\ominus,j} &= [\theta_{\ominus,j}, \theta_j - \varepsilon_{\ominus,j}], \\ I_{\oplus,j} &= [\theta_j + \varepsilon_{\oplus,j}, \theta_{\oplus,j}], \end{aligned} \tag{42}$$

where $\varepsilon_{\ominus,j}$ and $\varepsilon_{\oplus,j}$ are two sufficiently small positive real numbers, so that we also have

$$\begin{aligned} \theta_{\ominus,j} &< \theta_j - \varepsilon_{\ominus,j}, \\ \theta_j + \varepsilon_{\oplus,j} &< \theta_{\oplus,j}. \end{aligned} \tag{43}$$

Let us now consider sectional primitives of the real function $f_a(\theta)$ on these two intervals. Since the singularity of $f_a(\theta)$ at θ_j may not be integrable, we cannot integrate across the singularity, but we may integrate within these two lateral closed intervals, thus defining two sectional primitives of $f_a(\theta)$, one to the left and another one to the right of θ_j,

$$\begin{aligned} f_{a,\ominus}^{-1\prime}(\theta) &= \int_{\theta_{0,\ominus,j}}^{\theta} d\theta_\ominus\, f_a(\theta_\ominus), \\ f_{a,\oplus}^{-1\prime}(\theta) &= \int_{\theta_{0,\oplus,j}}^{\theta} d\theta_\oplus\, f_a(\theta_\oplus), \end{aligned} \tag{44}$$

where $f_a^{-1\prime}(\theta)$ is the notation for a primitive of $f_a(\theta)$ with respect to θ, where $\theta_{0,\ominus,j}$ and $\theta_{0,\oplus,j}$ are two arbitrary reference points, one within each of the two lateral closed intervals, and where we have

$$\begin{aligned} \theta_{\ominus,j} \leq &\quad \theta_\ominus \quad \leq \theta_j - \varepsilon_{\ominus,j}, \\ \theta_{\ominus,j} \leq &\quad \theta_{0,\ominus,j} \quad \leq \theta_j - \varepsilon_{\ominus,j}, \\ \theta_j + \varepsilon_{\oplus,j} \leq &\quad \theta_\oplus \quad \leq \theta_{\oplus,j}, \\ \theta_j + \varepsilon_{\oplus,j} \leq &\quad \theta_{0,\oplus,j} \quad \leq \theta_{\oplus,j}. \end{aligned} \tag{45}$$

If we change variables from θ to λ on the two sectional integrals in Equation (44), we get

$$\begin{aligned} f_{a,\ominus}^{-1\prime}(\theta) &= \int_{\lambda_{0,\ominus,j}}^{\lambda} d\lambda_\ominus\, \frac{d\theta}{d\lambda}(\lambda_\ominus)\, f_b(\lambda_\ominus), \\ f_{a,\oplus}^{-1\prime}(\theta) &= \int_{\lambda_{0,\oplus,j}}^{\lambda} d\lambda_\oplus\, \frac{d\theta}{d\lambda}(\lambda_\oplus)\, f_b(\lambda_\oplus), \end{aligned} \tag{46}$$

where $\lambda_{0,\ominus,j}$ and $\lambda_{0,\oplus,j}$ are the reference points on C_b corresponding respectively to $\theta_{0,\ominus,j}$ and $\theta_{0,\oplus,j}$. Since the derivative $d\theta/d\lambda$ which appears in these integrals is finite everywhere on C_b, and therefore limited, due to the fact that the inverse conformal transformation $\gamma^{(-1)}(z_b)$ is analytic and hence differentiable on the curve C_b, it follows that there are two real numbers R_m and R_M such that

$$R_m \leq \frac{d\theta}{d\lambda}(\lambda) \leq R_M \tag{47}$$

everywhere on C_b. Note that, since the conformal transformation, besides being continuous and differentiable, is also invertible on C_b, the derivative above cannot change sign and thus cannot be zero. Therefore, the two bounds R_m and R_M may be chose to have the same sign. By exchanging the derivative by these extreme values we can obtain upper and lower bounds for the sectional integrals, and therefore we get for the sectional primitives in Equation (46),

$$\begin{aligned} R_m \int_{\lambda_{0,\ominus,j}}^{\lambda} d\lambda_\ominus\, f_b(\lambda_\ominus) \;\leq\; f_{a,\ominus}^{-1\prime}(\theta) \;\leq\; R_M \int_{\lambda_{0,\ominus,j}}^{\lambda} d\lambda_\ominus\, f_b(\lambda_\ominus), \\ R_m \int_{\lambda_{0,\oplus,j}}^{\lambda} d\lambda_\oplus\, f_b(\lambda_\oplus) \;\leq\; f_{a,\oplus}^{-1\prime}(\theta) \;\leq\; R_M \int_{\lambda_{0,\oplus,j}}^{\lambda} d\lambda_\oplus\, f_b(\lambda_\oplus). \end{aligned} \tag{48}$$

We now recognize the integrals that appear in these expressions as the sectional primitives of the function $f_b(\lambda)$, so that we get

$$
\begin{aligned}
R_m\, f_{b,\ominus}^{-1\prime}(\lambda) &\leq f_{a,\ominus}^{-1\prime}(\theta) \leq R_M\, f_{b,\ominus}^{-1\prime}(\lambda), \\
R_m\, f_{b,\oplus}^{-1\prime}(\lambda) &\leq f_{a,\oplus}^{-1\prime}(\theta) \leq R_M\, f_{b,\oplus}^{-1\prime}(\lambda).
\end{aligned}
\tag{49}
$$

These expressions are true so long as $\varepsilon_{\ominus,j}$ and $\varepsilon_{\oplus,j}$, as well as the corresponding quantities $\delta_{\ominus,j}$ and $\delta_{\oplus,j}$ on C_b, are not zero, but since the singularities at $z_{b,j}$ are non-integrable hard ones, we cannot take the limit in which these quantities tend to zero. Note that, since R_m and R_M have the same sign, if the sectional primitives of the function $f_b(\lambda)$ diverge to infinity in this limit, then so do the sectional primitives of the function $f_a(\theta)$. Therefore, we may conclude that the hard singularity of $f_a(\theta)$ is also a non-integrable one. Since we may take further sectional integrals of these expressions, without affecting the inequalities, it is immediately apparent that, after a total of n successive piecewise integrations, we get for the $n^{\rm th}$ sectional primitives

$$
\begin{aligned}
R_m\, f_{b,\ominus}^{-n\prime}(\lambda) &\leq f_{a,\ominus}^{-n\prime}(\theta) \leq R_M\, f_{b,\ominus}^{-n\prime}(\lambda), \\
R_m\, f_{b,\oplus}^{-n\prime}(\lambda) &\leq f_{a,\oplus}^{-n\prime}(\theta) \leq R_M\, f_{b,\oplus}^{-n\prime}(\lambda).
\end{aligned}
\tag{50}
$$

Since the non-integrable hard singularity of $f_b(\lambda)$ at the point λ_j which corresponds to θ_j has a finite degree of hardness, according to the definition of the degrees of hardness, which was given in [1] and discussed in detail for the case of real functions in [4], there is a value of n such that the limit in which $\delta_{\ominus,j} \to 0$ and $\delta_{\oplus,j} \to 0$ can be taken for the sectional primitives $f_{b,\ominus}^{-n\prime}(\lambda)$ and $f_{b,\oplus}^{-n\prime}(\lambda)$, thus implying that the $n^{\rm th}$ piecewise primitive $f_b^{-n\prime}(\lambda)$ of $f_b(\lambda)$ is an integrable real function on the whole interval $[\lambda_{\ominus,j}, \lambda_{\oplus,j}]$ that corresponds to $[\theta_{\ominus,j}, \theta_{\oplus,j}]$, with a borderline hard singularity, with degree of hardness zero, at the point λ_j. It follows from the inequalities, therefore, that the corresponding limit in which $\varepsilon_{\ominus,j} \to 0$ and $\varepsilon_{\oplus,j} \to 0$ can be taken for the functions $f_{a,\ominus}^{-n\prime}(\theta)$ and $f_{a,\oplus}^{-n\prime}(\theta)$, thus implying that the $n^{\rm th}$ piecewise primitive $f_a^{-n\prime}(\theta)$ of $f_a(\theta)$ is also an integrable real function on the whole interval $[\theta_{\ominus,j}, \theta_{\oplus,j}]$, with a borderline hard singularity, with degree of hardness zero, at θ_j. Therefore, the non-integrable hard singularity of $f_a(\theta)$ at θ_j has a finite degree of hardness, to wit the same degree of hardness n of the corresponding non-integrable hard singularity of $f_b(\lambda)$ at λ_j. This establishes Lemma 4.

We have therefore established that, so long as $f_b(\lambda)$ is locally integrable almost everywhere on C_b, and so long as its non-integrable hard singularities have finite degrees of hardness, these same two facts are true for $f_a(\theta)$ on C_a. Since we thus see that the necessary properties of the real functions are preserved by the conformal transformation, we are therefore in a position to use the result of Theorem 2 in Section 3 in order to extend the existence theorem of the Dirichlet problem to non-integrable real functions which are, however, integrable almost everywhere on C_b, still for the case of differentiable curves.

In this section, using once again the results from the previous sections, we will establish the following theorem.

Theorem 5: *Given a differentiable simple closed curve C of finite total length on the complex plane $z = x + \imath y$, given the invertible conformal transformation $\gamma(z)$ whose derivative has no zeros on the closed unit disk, that maps it from the unit circle, and given a real function $f(\lambda)$ on that curve, that satisfies the list of conditions described below, there is a solution $u(x, y)$ of the Dirichlet problem of the Laplace equation within the interior of that curve, that assumes the given values $f(\lambda)$ almost everywhere at the curve.*

Proof 5.1:

Similarly to what was done in the two previous cases, in Sections 5 and 6, the proof consists of using the conformal transformation between the closed unit disk and the union of the curve C with its interior, which according to the analysis in Section 4 always exists, to map the given boundary condition on C onto a corresponding boundary condition on the unit circle, then using the proof of existence established before by Theorem 2 in Section 3 for the closed unit disk to establish the existence of the solution of the corresponding Dirichlet problem on that disk, and finally using once more the conformal transformation to map the resulting solution back to C and its interior, thus obtaining the solution of the original Dirichlet problem. The list of conditions on the real functions is now the following.

1. The real function $f(\lambda)$ is locally integrable almost everywhere on C, including the cases in which this function is globally integrable there.

2. The number of hard singularities on the unit circle of the corresponding inner analytic function $w(z)$ on the unit disk is finite.

3. The hard singularities of the corresponding inner analytic function $w(z)$ have finite degrees of hardness.

The rest of the proof is identical to that of the two previous cases, in Sections 5 and 6. Therefore, once again we may conclude that, due to the existence theorem of the Dirichlet problem on the unit disk of the plane z_a, which in this case was established by Theorem 2 in Section 3, we know that there is an inner analytic function $w_a(z_a)$ such that its real part $u_a(\rho, \theta)$ is harmonic within the open unit disk and satisfies $u_a(1, \theta) = f_a(\theta)$ almost everywhere at the boundary C_a. Just as before, we get on the z_b plane the complex function $w_b(z_b)$ which is analytic in the interior of the curve C_b. Therefore, the real part $u_b(x, y)$ of $w_b(z_b)$ is harmonic and thus satisfies

$$\nabla^2 u_b(x, y) = 0 \tag{51}$$

in the interior of C_b, while we also have that

$$u_b(x, y) = f_b(\lambda), \tag{52}$$

almost everywhere on C_b. This establishes the existence, by construction, of the solution of the Dirichlet problem on the z_b plane, under our current hypotheses. This completes the proof of Theorem 5.

In this way we have generalized the proof of existence of the Dirichlet problem from the unit circle to all differentiable simple closed curves with finite total lengths on the plane, for boundary conditions given by non-integrable real functions which are locally integrable almost everywhere and have at most a finite set of hard singular points.

8 Non-Integrable Functions on Non-Differentiable Curves

In this section we will show how one can generalize the proof of existence of the Dirichlet problem on the unit disk, given by Theorem 2 in Section 3, to the case in which we have, as the boundary condition, non-integrable real functions $f_b(\lambda)$ defined on boundaries given by non-differentiable curves C_b on the plane. We will be able to do this if the non-integrable real functions, despite being non-integrable over the whole curves C_b, are locally integrable almost everywhere on those curves, and if, in addition to this, the non-integrable hard singularities involved have finite degrees of hardness. The definition of the concept of local integrability almost everywhere is that given by Definition 4, in Section 7. The additional difficulty that appears in this case is the same which was discussed in Section 6, due to the fact that the curve C_b is not differentiable at the finite set of points $z_{b,i}$, for $i \in \{1, \ldots, N\}$. Just as in that case, this will require that we impose one additional limitation on the real functions giving the boundary conditions, namely that any integrable hard singularities where they diverge to infinity do not coincide with any of the points $z_{b,i}$.

The proof will follow the general lines of the one given for integrable real functions in Section 6, with the difference that, just as we did in Section 7, instead of showing that the corresponding functions $f_a(\theta)$ on the unit circle C_a are integrable there, we will show that they are locally integrable almost everywhere there. In addition to this, instead of using the existence theorem for integrable real function on the unit circle, which was given by Theorem 1 in Section 2, we will use the corresponding result for non-integrable real functions which are locally integrable almost everywhere on the unit circle, which was given by Theorem 2 in Section 3. Since that result depends also on the non-integrable hard singularities of the real functions having finite degrees of hardness, we will also show that the hypothesis that the functions $f_b(\lambda)$ have this property implies that the corresponding functions $f_a(\theta)$ have the same property as well. In order to do this we will use once again the technique of piecewise integration which was introduced in [4].

The preliminary result given by Lemma 3 in Section 7 is still valid here. As a consequence of this we may conclude at once that, under the conditions that we have here, the hypothesis that $f_b(\lambda)$ is locally integrable almost everywhere on C_b, with the exclusion of the finite set of points $z_{b,j}$, for $j \in \{1, \ldots, M\}$, implies that $f_a(\theta)$ is locally integrable almost everywhere on the unit circle C_a, with the exclusion of the corresponding finite set of points $z_{a,j} = \gamma^{(-1)}(z_{b,j})$.

We must now discuss the issue of the degrees of hardness of the non-integrable hard singularities of the function $f_a(\theta)$ on C_a. Since by hypothesis $f_b(\lambda)$ has non-integrable hard singularities at the points $z_{b,j}$, it clearly follows that $f_a(\theta)$ also has hard singularities at the corresponding points $z_{a,j}$, which may be non-integrable ones. In order to discuss their degrees of hardness we will use the technique of piecewise integration, that is, we will consider sectional integrals of $f_a(\theta)$ on closed intervals contained within a neighborhood of the point $z_{a,j}$ where it has a single isolated hard singularity. Let us show the following preliminary fact about a real function $f_a(\theta)$ defined from a real function $f_b(\lambda)$ which is locally integrable almost everywhere on C_b, and which has non-integrable hard singularities at the finite set of points $z_{b,j}$.

Lemma 5: *Given a real function $f_b(\lambda)$ which has an isolated non-integrable hard singularity with finite degree of hardness at a point $z_{b,j}$ on C_b, and which is such that the hard singularities where it diverges to infinity do not coincide with any of the points $z_{b,i}$ where C_b is non-differentiable, it follows that the corresponding function $f_a(\theta)$ defined on the unit circle C_a by $f_a(\theta) = f_b(\lambda)$ has an isolated non-integrable hard singularity with finite degree of hardness at the corresponding point $z_{a,j}$ on C_a.*

Since the real functions must diverge to infinity at non-integrable hard singular points, the fact that $f_a(\theta)$ has an isolated hard singularity on $z_{a,j}$ is immediate. Since these singularities are all isolated from each other, and since they do not coincide with the points $z_{b,i}$ where C_b is non-differentiable, there is on C_b a neighborhood of the point $z_{b,j}$ within which C_b is differentiable and there are no other non-integrable hard singularities of $f_b(\lambda)$. Since the conformal mapping is continuous, it follows that there is on C_a a neighborhood of the corresponding point $z_{a,j}$ within which there are no zeros of the derivative of $\gamma(z_a)$ and no other hard singularities of $f_a(\theta)$. The construction of the two sectional primitives of $f_a(\theta)$ by means of piecewise integration is the same as the one which was executed before for Lemma 4 in Section 7, resulting in Equation (44). After we change variables from θ to λ on the two sectional integrals in that equation we get

$$\begin{aligned} f_{a,\ominus}^{-1\prime}(\theta) &= \int_{\lambda_{0,\ominus,j}}^{\lambda} d\lambda_\ominus \, \frac{d\theta}{d\lambda}(\lambda_\ominus)\, f_b(\lambda_\ominus), \\ f_{a,\oplus}^{-1\prime}(\theta) &= \int_{\lambda_{0,\oplus,j}}^{\lambda} d\lambda_\oplus \, \frac{d\theta}{d\lambda}(\lambda_\oplus)\, f_b(\lambda_\oplus), \end{aligned} \tag{53}$$

where all the symbols involved are the same as before. Since the derivative $d\theta/d\lambda$ which appears in these integrals is finite, and therefore limited, everywhere within the two lateral intervals involved, due to the fact that the inverse conformal transformation $\gamma^{(-1)}(z_b)$ is analytic and therefore differentiable within these intervals, it follows that there are two pairs of real numbers $R_{\ominus,m}$ and $R_{\ominus,M}$, as well as $R_{\oplus,m}$ and $R_{\oplus,M}$, such that

$$\begin{aligned} R_{\ominus,m} \;&\leq\; \frac{d\theta}{d\lambda}(\lambda_\ominus) \;\leq\; R_{\ominus,M}, \\ R_{\oplus,m} \;&\leq\; \frac{d\theta}{d\lambda}(\lambda_\oplus) \;\leq\; R_{\oplus,M}, \end{aligned} \tag{54}$$

everywhere within each interval. Note that, since the conformal transformation, besides being continuous and differentiable, is also invertible within each interval, the derivatives above cannot change sign and thus cannot be zero. Therefore, the pair of bounds $R_{\ominus,m}$ and $R_{\ominus,M}$ may be chosen to have the same sign, and so may the pair of bounds $R_{\oplus,m}$ and $R_{\oplus,M}$. By exchanging the derivative by these extreme values we can obtain upper and lower bounds for the sectional integrals, and therefore we get for the sectional primitives in Equation (53),

$$\begin{aligned} R_{\ominus,m} \int_{\lambda_{0,\ominus,j}}^{\lambda} d\lambda_\ominus \, f_b(\lambda_\ominus) \;&\leq\; f_{a,\ominus}^{-1\prime}(\theta) \;\leq\; R_{\ominus,M} \int_{\lambda_{0,\ominus,j}}^{\lambda} d\lambda_\ominus \, f_b(\lambda_\ominus), \\ R_{\oplus,m} \int_{\lambda_{0,\oplus,j}}^{\lambda} d\lambda_\oplus \, f_b(\lambda_\oplus) \;&\leq\; f_{a,\oplus}^{-1\prime}(\theta) \;\leq\; R_{\oplus,M} \int_{\lambda_{0,\oplus,j}}^{\lambda} d\lambda_\oplus \, f_b(\lambda_\oplus). \end{aligned} \tag{55}$$

We now recognize the integrals that appear in these expressions as the sectional primitives of the function $f_b(\lambda)$, so that we get

$$\begin{aligned} R_{\ominus,m}\, f_{b,\ominus}^{-1\prime}(\lambda) \;&\leq\; f_{a,\ominus}^{-1\prime}(\theta) \;\leq\; R_{\ominus,M}\, f_{b,\ominus}^{-1\prime}(\lambda), \\ R_{\oplus,m}\, f_{b,\oplus}^{-1\prime}(\lambda) \;&\leq\; f_{a,\oplus}^{-1\prime}(\theta) \;\leq\; R_{\oplus,M}\, f_{b,\oplus}^{-1\prime}(\lambda). \end{aligned} \tag{56}$$

These expressions are true so long as $\varepsilon_{\ominus,j}$ and $\varepsilon_{\oplus,j}$, as well as the corresponding quantities $\delta_{\ominus,j}$ and $\delta_{\oplus,j}$ on C_b, are not zero, but since the singularities at $z_{b,j}$ are non-integrable hard ones, we cannot take the limit in which these quantities tend to zero. Note that, since $R_{\ominus,m}$ and $R_{\ominus,M}$ have the same sign, and also $R_{\oplus,m}$ and $R_{\oplus,M}$ have the same sign, if the sectional primitives of the function $f_b(\lambda)$ diverge to infinity in this limit, then so do the sectional primitives of the function $f_a(\theta)$. Therefore, we may conclude that the hard singularity of $f_a(\theta)$ is also a non-integrable one. Since we may take further sectional integrals of these expressions, without affecting the inequalities, it is immediately apparent that, after a total of n successive piecewise integrations, we get for the n^{th} sectional primitives

$$\begin{aligned} R_{\ominus,m}\, f_{b,\ominus}^{-n\prime}(\lambda) \quad &\leq \quad f_{a,\ominus}^{-n\prime}(\theta) \quad \leq \quad R_{\ominus,M}\, f_{b,\ominus}^{-n\prime}(\lambda), \\ R_{\oplus,m}\, f_{b,\oplus}^{-n\prime}(\lambda) \quad &\leq \quad f_{a,\oplus}^{-n\prime}(\theta) \quad \leq \quad R_{\oplus,M}\, f_{b,\oplus}^{-n\prime}(\lambda). \end{aligned} \tag{57}$$

Since the non-integrable hard singularity of $f_b(\lambda)$ at the point λ_j which corresponds to θ_j has a finite degree of hardness, according to the definition of the degrees of hardness, which was given in [1] and discussed in detail for the case of real functions in [4], there is a value of n such that the limit in which $\delta_{\ominus,j} \to 0$ and $\delta_{\oplus,j} \to 0$ can be taken for the sectional primitives $f_{b,\ominus}^{-n\prime}(\lambda)$ and $f_{b,\oplus}^{-n\prime}(\lambda)$, thus implying that the $n^{\rm th}$ piecewise primitive $f_b^{-n\prime}(\lambda)$ of $f_b(\lambda)$ is an integrable real function on the whole interval $[\lambda_{\ominus,j}, \lambda_{\oplus,j}]$, with a borderline hard singularity, with degree of hardness zero, at the point λ_j. It follows from the inequalities, therefore, that the corresponding limit in which $\varepsilon_{\ominus,j} \to 0$ and $\varepsilon_{\oplus,j} \to 0$ can be taken for the functions $f_{a,\ominus}^{-n\prime}(\theta)$ and $f_{a,\oplus}^{-n\prime}(\theta)$, thus implying that the $n^{\rm th}$ piecewise primitive $f_a^{-n\prime}(\theta)$ of $f_a(\theta)$ is also an integrable real function on the whole interval $[\theta_{\ominus,j}, \theta_{\oplus,j}]$, with a borderline hard singularity, with degree of hardness zero, at θ_j. Therefore, the non-integrable hard singularity of $f_a(\theta)$ at θ_j has a finite degree of hardness, to wit the same degree of hardness n of the corresponding non-integrable hard singularity of $f_b(\lambda)$ at λ_j. This establishes Lemma 5.

We have therefore established that, under the hypothesis that the hard singularities where $f_b(\lambda)$ diverges to infinity do not coincide with any of the points $z_{b,i}$ where C_b is non-differentiable, so long as $f_b(\lambda)$ is locally integrable almost everywhere on C_b, and so long as its non-integrable hard singularities have finite degrees of hardness, these same two facts are true for $f_a(\theta)$ on C_a. Since we thus see that the necessary properties of the real functions are preserved by the conformal transformation, we are therefore in a position to use the result of Theorem 2 in Section 3 in order to extend the existence theorem of the Dirichlet problem to non-integrable real functions which are, however, integrable almost everywhere on C_b, this time for the case of non-differentiable curves.

In this section, using one more time the results from the previous sections, we will establish the following theorem.

Theorem 6: *Given a simple closed curve C of finite total length on the complex plane $z = x + \imath y$, which is non-differentiable at a given finite set of points z_i, for $i \in \{1, \ldots, N\}$, given the conformal transformation $\gamma(z)$ that maps it from the unit circle, whose derivative has zeros on the unit circle at the corresponding set of points, and given a real function $f(\lambda)$ on that curve, that satisfies the list of conditions described below, there is a solution $u(x, y)$ of the Dirichlet problem of the Laplace equation within the interior of that curve, that assumes the given values $f(\lambda)$ almost everywhere at the curve.*

Proof 6.1:

Similarly to what was done in the three previous sections, the proof consists of using the conformal transformation between the closed unit disk and the union of the curve C with its interior, which according to the analysis in Section 4 always exists, to map the given boundary condition on C onto a corresponding boundary condition on the unit circle, then using the proof of existence established before by Theorem 2 in Section 3 for the closed unit disk to establish the existence of the solution of the corresponding Dirichlet problem on that disk, and finally using once more the conformal transformation to map the resulting solution back to C and its interior, thus obtaining the solution of the original Dirichlet problem. The list of conditions on the real functions is now the following.

1. The real function $f(\lambda)$ is locally integrable almost everywhere on C, including the cases in which this function is globally integrable there.

2. The number of hard singularities on the unit circle of the corresponding inner analytic function $w(z)$ on the unit disk is finite.

3. The hard singularities of the corresponding inner analytic function $w(z)$ have finite degrees of hardness.

4. The hard singularities of $f(\lambda)$ where it diverges to infinity are not located at any of the points where the curve C is non-differentiable.

The rest of the proof is identical to that of the three previous cases. Therefore, once again we may conclude that, due to the existence theorem of the Dirichlet problem on the unit disk of the plane z_a, which in this case was established in by Theorem 2 Section 3, we know that there is an inner analytic function $w_a(z_a)$ such that its real part $u_a(\rho, \theta)$ is harmonic within the open unit disk and satisfies $u_a(1, \theta) = f_a(\theta)$ almost everywhere at the boundary C_a. Just as before, we get on the z_b plane the complex function $w_b(z_b)$ which is analytic in the interior of the curve C_b. Therefore, the real part $u_b(x, y)$ of $w_b(z_b)$ is harmonic and thus satisfies

$$\nabla^2 u_b(x, y) = 0 \tag{58}$$

in the interior of C_b, while we also have that

$$u_b(x, y) = f_b(\lambda), \tag{59}$$

almost everywhere on C_b. This establishes the existence, by construction, of the solution of the Dirichlet problem on the z_b plane, under our current hypotheses. This completes the proof of Theorem 6.

In this way we have generalized the proof of existence of the Dirichlet problem from the unit circle to all simple closed curves with finite total lengths on the plane, that can be either differentiable or non-differentiable on a finite set of points, but now for boundary conditions given by non-integrable real functions which are locally integrable almost everywhere and have at most a finite set of hard singular points.

9 Conclusions and Outlook

A very general proof of the existence of the solution of the Dirichlet boundary value problem of the Laplace equation on the plane was presented. The proof is valid not only for a very large class of real functions at the boundary, but also for a large class of boundary curves, with and without points of non-differentiability. The proof was presented in incremental steps, each generalizing the previous ones. The proofs for the unit circle are based on the complex-analytic structure within the unit disk presented and developed in previous papers [1–4]. The generalization for curves other than the unit circle uses the conformal mapping results associated to the famous Riemann mapping theorem. The most general statement of the theorem established here reads as follows.

Given a real function $f(\lambda)$ that defines the boundary condition on a plane curve C parametrized by the real arc-length variable λ, so long as the real function is locally integrable almost everywhere on C, and is such that the corresponding inner analytic function has at most a finite number of hard singularities with finite degrees of hardness, so long as C is a simple closed curve with finite total length and at most a finite number of points of non-differentiability, and so long as the hard singular points of $f(\lambda)$ where it diverges to infinity do not coincide with any of the points where the curve is not differentiable, there exists a real function $u(x,y)$ that satisfies $\nabla^2 u(x,y) = 0$ in the interior of C and that satisfies $u(x,y) = f(\lambda)$ almost everywhere on C.

The proof is constructive, and consists of constructing from $f(\lambda)$ an analytic function in the interior of C, of which $u(x,y)$ is the real part. The theorem is quite general, including large classes of both boundary conditions and boundary curves.

Further extensions of the theorem may be possible. For example, the proofs established in Sections 2 and 3 can be rather trivially extended to include as well the whole space of singular Schwartz distributions discussed in [2], that is, they can be extended to generalized real functions. This allows one to discuss some rather unusual Dirichlet problems in which the boundary condition is given by a singular real object such as the Dirac delta "function" or its derivatives. As mentioned in [1], a possible further extension would be to real functions with a countable infinity of hard singular points which have, however, a finite number of accumulation points. The requirement that the hard singular points of $f(\lambda)$ where it diverges to infinity do not coincide with the points where the curve C is non-differentiable seems to be a technical quirk, and probably can be eliminated. It is important to note that the proof is intrinsically limited to two-dimensional problems on the plane.

It is interesting to observe that the uniqueness of the solution can also be discussed in this context, in terms of the fact that $f(\theta) \equiv 0$ corresponds to the Fourier coefficients $\alpha_0 = 0$, $\alpha_k = 0$ and $\beta_k = 0$, for all k, and therefore to the complex Taylor coefficients $c_0 = 0$ and $c_k = 0$, for all k, and therefore to the identically zero inner analytic function $w(z) \equiv 0$. Given an integrable real function $f(\theta)$ and two corresponding solutions $w_1(z)$ and $w_2(z)$ of the Dirichlet problem, we simply consider $w(z) = w_2(z) - w_1(z)$, which is therefore a solution of the Dirichlet problem with $f(\theta) \equiv 0$, and thus by construction is $w(z) \equiv 0$. It follows that $w_2(z) \equiv w_1(z)$, so that the solution is unique, in the sense that $u_2(\rho,\theta) = u_1(\rho,\theta)$ almost everywhere on the unit disk. We can say, in fact, that these two functions are equal at all points on the unit circle where they are well defined.

With some more work towards its generalization, the result presented here points, perhaps, to an even more general result, according to which the solution of the Dirichlet problem of the Laplace equation in two dimensions, in essence, *always* exists, in the sense that it exists under all conceivable circumstances in which it makes any sense at all to pose the corresponding boundary value problem. Already, even with the result as it is now, this is almost the case in what concerns the applications to Physics.

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