Complex Analysis of Real Functions

IV: Non-Integrable Real Functions

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Abstract

In the context of the complex-analytic structure within the unit disk centered at the origin of the complex plane, that was presented in a previous paper, we show that a certain class of non-integrable real functions can be represented within that same structure. In previous papers it was shown that essentially all integrable real functions, as well as all singular Schwartz distributions, can be represented within that same complex-analytic structure. The large class of non-integrable real functions which we analyze here can therefore be represented side by side with those other real objects, thus allowing all these objects to be treated in a unified way.

1 Introduction

In a previous paper [1] we introduced a certain complex-analytic structure within the unit disk of the complex plane, and showed that one can represent essentially all integrable real functions within that structure. In another, subsequent previous paper [2], we showed that one can also represent within the same structure the singular objects known as distributions, loosely in the sense of the Schwartz theory of distributions [3], which are also known as generalized real functions. In this paper we will show that a large class of non-integrable real functions can also be represented within that same structure. All these objects will be interpreted as parts of this larger complex-analytic structure, within which they can be treated and manipulated in a robust and unified way.

The construction presented in [1], which leads to the inclusion of all integrable real functions in the complex-analytic structure, starts from the Fourier coefficients of an arbitrarily given real function on the unit circle. Since these coefficients are defined as integrals involving the real function, there is the obvious necessity that these real functions be integrable on that circle. Therefore, that construction will not work for functions which fail to be integrable there. However, once we are in the space of inner analytic functions within the open unit disk, the concept of integral-differential chains of inner analytic functions, which was introduced in [1], comes to our rescue. It will allow us to extend the representation within the complex-analytic structure to a large class of non-integrable real functions.

For ease of reference, we include here a one-page synopsis of the complex-analytic structure introduced in [1]. It consists of certain elements within complex analysis [4], as well as of their main properties.

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Synopsis: The Complex-Analytic Structure

An *inner analytic function* $w(z)$ is simply a complex function which is analytic within the open unit disk. An inner analytic function that has the additional property that $w(0) = 0$ is a *proper inner analytic function.* The *angular derivative* of an inner analytic function is defined by

$$w^{\bullet}(z) = \imath z \, \frac{dw(z)}{dz}. \tag{1}$$

By construction we have that $w^{\bullet}(0) = 0$, for all $w(z)$. The *angular primitive* of an inner analytic function is defined by

$$w^{-1^{\bullet}}(z) = -\imath \int_{0}^{z} dz' \, \frac{w(z') - w(0)}{z'}. \tag{2}$$

By construction we have that $w^{-1^{\bullet}}(0) = 0$, for all $w(z)$. In terms of a system of polar coordinates (ρ, θ) on the complex plane, these two analytic operations are equivalent to differentiation and integration with respect to θ, taken at constant ρ. These two operations stay within the space of inner analytic functions, they also stay within the space of proper inner analytic functions, and they are the inverses of one another. Using these operations, and starting from any proper inner analytic function $w^{0^{\bullet}}(z)$, one constructs an infinite *integral-differential chain* of proper inner analytic functions,

$$\left\{ \ldots, w^{-3^{\bullet}}(z), w^{-2^{\bullet}}(z), w^{-1^{\bullet}}(z), w^{0^{\bullet}}(z), w^{1^{\bullet}}(z), w^{2^{\bullet}}(z), w^{3^{\bullet}}(z), \ldots \right\}. \tag{3}$$

Two different such integral-differential chains cannot ever intersect each other. There is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, namely the null chain, in which all members are the null function $w(z) \equiv 0$.

A general scheme for the classification of all possible singularities of inner analytic functions is established. A singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity.* Angular integration takes soft singularities to other soft singularities, and angular differentiation takes hard singularities to other hard singularities.

Gradations of softness and hardness are then established. A hard singularity that becomes a soft one by means of a single angular integration is a *borderline hard* singularity, with degree of hardness zero. The *degree of softness* of a soft singularity is the number of angular differentiations that result in a borderline hard singularity, and the *degree of hardness* of a hard singularity is the number of angular integrations that result in a borderline hard singularity. Singularities which are either soft or borderline hard are integrable ones. Hard singularities which are not borderline hard are non-integrable ones.

Given an integrable real function $f(\theta)$ on the unit circle, one can construct from it a unique corresponding inner analytic function $w(z)$. Real functions are obtained through the $\rho \to 1_{(-)}$ limit of the real and imaginary parts of each such inner analytic function and, in particular, the real function $f(\theta)$ is obtained from the real part of $w(z)$ in this limit. The pair of real functions obtained from the real and imaginary parts of one and the same inner analytic function are said to be mutually Fourier-conjugate real functions.

Singularities of real functions can be classified in a way which is analogous to the corresponding complex classification. Integrable real functions are typically associated with inner analytic functions that have singularities which are either soft or at most borderline hard. This ends our synopsis.

When we discuss real functions in this paper, some properties will be globally assumed for these functions, just as was done in the previous papers [1, 2, 5] leading to this one. These are rather weak conditions to be imposed on these functions, that will be in force throughout this paper. It is to be understood, without any need for further comment, that these conditions are valid whenever real functions appear in the arguments. These weak conditions certainly hold for any real functions that are obtained as restrictions of corresponding inner analytic functions to the unit circle.

The most basic condition is that the real functions must be measurable in the sense of Lebesgue, with the usual Lebesgue measure [6, 7]. The second global condition we will impose is that the functions have no removable singularities. The third and last global condition is that the number of hard singularities on the unit circle be finite, and hence that they be all isolated from one another. There will be no limitation on the number of soft singularities.

The material contained in this paper is a development, reorganization and extension of some of the material found, sometimes still in rather rudimentary form, in the papers [8–12].

2 Preliminary Considerations

Here we will quickly review some results of previous papers in this series, which are necessary for the work currently at hand. We will also establish some equally necessary groundwork for the discussion that follows in Sections 3 and 4.

2.1 Singularities of Real Functions

In this paper we will discuss non-integrable real functions, which therefore have on the unit circle hard singularities which are not just borderline hard ones. It is therefore necessary to consider in greater detail the issue of the singularities of real functions. Therefore, let us discuss the translation of the classification of singularities, which has been established in [1] for inner analytic functions in the unit disk of the complex plane, to the case of real functions on the unit circle.

First of all, let us discuss the concept of a removable singularity. This is a well-known concept for analytic functions in the complex plane. What we mean by a removable singularity in the case of real functions on the unit circle is a singular point such that both lateral limits of the function to that point exist and result in the same finite real value, but where the function has been arbitrarily defined to have some other finite real value. This is therefore a point were the function can be redefined by continuity, resulting in a continuous function at that point. These are, therefore, trivial singularities, which we will simply rule out of our discussions in this paper.

The concept of a singularity is the same, namely a point where the function is not analytic. The concepts of soft and hard singularities are carried in a straightforward way from the case of complex functions to that of real functions. The only significant difference is that the concept of the limit of the function to a point is now taken to be the real one, along the unit circle. The existence of the limit is a weaker condition in the real case, because in the complex case the limit must exist on and be independent of the continuum of directions along which one can take it along the complex plane, while in the real case the limit only has to exist and be the same along the two lateral directions. This means that, if one takes a path across a singularity of an analytic function in order to define a real function, the singularity of the real function may be soft even if the singularity of the analytic function is hard, and the singularity of the real function may be integrable even if the singularity of the analytic function is not.

At this point it is interesting to note that it might be useful to consider classifying all inner analytic functions $w(z) = u(\rho,\theta) + \imath v(\rho,\theta)$ according to whether or not they have the same basic analytic properties, taken in the complex sense, as the corresponding real functions $u(1,\theta)$, taken in the real sense. One could say that a *regular* inner analytic function is one that has, at all its singular points on the unit circle, the same status as the corresponding real function, regarding the most fundamental analytic properties. By contrast, an *irregular* inner analytic function would be one that fails to have the same status as the corresponding real function, regarding one or more of the these analytic properties, such as those of integrability, continuity, and a given level of multiple differentiability.

The gradations corresponding to soft and hard singularities can be implemented for real functions in terms of the integrability or non-integrability of the singularities. The soft singularities are all integrable, and a singularity which is both hard and integrable is necessarily a borderline hard singularity, that can be immediately associated to the degree of hardness zero. The degree of softness of an isolated soft singularity is the number of differentiations with respect to θ of the real function $f(\theta)$, in a neighborhood of the singular point, which are required for the singularity to become a borderline hard one. Observing that a soft singularity of degree one means that the function is continuous but not differentiable, an alternative definition is that the degree of softness is one plus the number of differentiations with respect to θ of the real function $f(\theta)$, in a neighborhood of the singular point, which are required for the function to become non-differentiable at that point. The remaining problem is that of associating a degree of hardness to non-integrable hard singularities. This is the most important part of this structure in regard to our work in this paper, so let us examine it in more detail.

Let us assume that the real function $f(\theta)$ has an *isolated* non-integrable singular point at θ_1, which is therefore a hard singularity. What we mean here by this singularity being isolated is that the function has no other non-integrable hard singularities in a neighborhood of that point, and is thus integrable in each one of the two sides of that neighborhood. Consider then two points within that neighborhood on the unit circle, say $\theta_\ominus$ and $\theta_\oplus$, one to the left and the other to the right of θ_1, so that we have $\theta_\ominus < \theta_1 < \theta_\oplus$. It follows, therefore, that the function $f(\theta)$ is an integrable real function on the two closed intervals $[\theta_\ominus, \theta_1 - \varepsilon_\ominus]$ and $[\theta_1 + \varepsilon_\oplus, \theta_\oplus]$, where $\varepsilon_\ominus$ and $\varepsilon_\oplus$ are any sufficiently small strictly positive real numbers, such that $\theta_\ominus < \theta_1 - \varepsilon_\ominus$ and $\theta_1 + \varepsilon_\oplus < \theta_\oplus$. Therefore, we can integrate the function $f(\theta)$ within these two intervals, starting at some arbitrary reference points $\theta_{0,\ominus}$ and $\theta_{0,\oplus}$ strictly within each interval, defining in this way a pair of sectional primitives $f_\ominus^{-1\prime}(\theta)$ and $f_\oplus^{-1\prime}(\theta)$, one within each interval,

$$\begin{aligned} f_\ominus^{-1\prime}(\theta) &= \int_{\theta_{0,\ominus}}^{\theta} d\theta'\, f(\theta'), \\ f_\oplus^{-1\prime}(\theta) &= \int_{\theta_{0,\oplus}}^{\theta} d\theta'\, f(\theta'), \end{aligned} \tag{4}$$

where we have that both the argument θ and the reference points $\theta_{0,\ominus}$ and $\theta_{0,\oplus}$ are within the corresponding closed integration intervals. Therefore, on the left interval we have that $\theta_\ominus \leq \theta \leq \theta_1 - \varepsilon_\ominus$ and that $\theta_\ominus \leq \theta_{0,\ominus} \leq \theta_1 - \varepsilon_\ominus$, while on the right interval we have that $\theta_1 + \varepsilon_\oplus \leq \theta \leq \theta_\oplus$ and that $\theta_1 + \varepsilon_\oplus \leq \theta_{0,\oplus} \leq \theta_\oplus$. Note that, since $f(\theta)$ is integrable on those closed intervals, it follows that the piecewise primitive $f^{-1\prime}(\theta)$ formed by the pair of sectional primitives $f_\ominus^{-1\prime}(\theta)$ and $f_\oplus^{-1\prime}(\theta)$ is limited on those same closed intervals, and therefore is also integrable on them. Therefore, this process of sectional integration of the functions can be iterated indefinitely. Let us assume that we iterate this process n times, thus obtaining the n-fold primitive $f^{-n\prime}(\theta)$.

In general one cannot take the limits $\varepsilon_{\ominus} \to 0$ or $\varepsilon_{\oplus} \to 0$, because the singularity at θ_1 is a non-integrable one and therefore the limits of the asymptotic integrals on either side will diverge. However, if there is a number n of successive sectional integrations such that the resulting primitive $f^{-n\prime}(\theta)$ has a borderline hard singularity at θ_1, being therefore integrable on the whole closed interval $[\theta_{\ominus}, \theta_{\oplus}]$, then the two limits $\varepsilon_{\ominus} \to 0$ and $\varepsilon_{\oplus} \to 0$ can both be taken. In this case we may say that the hard singularity of $f(\theta)$ at θ_1 has degree of hardness n. In Section 4 we will have the opportunity to use these ideas regarding sectional integration on closed intervals that avoid non-integrable singular points.

2.2 Piecewise Polynomial Real Functions

In [2] we introduced the concept of piecewise polynomial real functions, in the context of the discussion of singular Schwartz distributions. These are the integrable real functions that are obtained on the integration side of the integral-differential chains to which the singular distributions belong, the latter being on the differentiation side of the same chains, with respect to the central position of the delta "function" itself. This concept will return to the discussion in this paper, where it will play an essential role. Therefore, let us review the definitions and the relevant results established in that previous paper.

Let us assume that we have a real function which is defined by polynomials in sections of the unit circle. These sections are open intervals separated by a finite set of points which are singularities of that real function. Let there be $N \geq 1$ singular points $\{\theta_1, \ldots, \theta_N\}$. It follows that in this case we must separate the unit circle into a set of N contiguous sections, consisting of open intervals between the singular points, that can be represented as the sequence

$$\left\{(\theta_1, \theta_2), \ldots, (\theta_{i-1}, \theta_i), (\theta_i, \theta_{i+1}), \ldots, (\theta_N, \theta_1)\right\}, \tag{5}$$

where we see that the sequence goes around the unit circle, and where we adopt the convention that each section (θ_i, θ_{i+1}) is numbered after the singular point θ_i at its left end.

In [2] we established a general notation for these piecewise polynomial real functions, as well as a formal definition for them. Here is the definition: given a real function $f_{(n)}(\theta)$ that is defined in a piecewise fashion by polynomials in $N \geq 1$ sections of the unit circle, with the exclusion of a finite non-empty set of N singular points θ_i, with $i \in \{1, \ldots, N\}$, so that the polynomial $P_i^{(n_i)}(\theta)$ at the $i^{\rm th}$ section has order n_i, we denote the function by

$$f_{(n)}(\theta) = \left\{P_i^{(n_i)}(\theta), i \in \{1, \ldots, N\}\right\}, \tag{6}$$

where n is the largest order among all the N orders n_i. We say that $f_{(n)}(\theta)$ is a *piecewise polynomial real function* of order n. Note that, being made out of finite sections of polynomials, the real function $f_{(n)}(\theta)$ is always an integrable real function.

These functions have some important properties, which were established in [2]. First and foremost, if the integrable real function $f_{(n)}(\theta)$ is a non-zero piecewise polynomial function of order n, then and only then its derivative $f_{(n)}^{(n+1)\prime}(\theta)$ with respect to θ is a superposition of a non-empty set of delta "functions" and derivatives of delta "functions" on the unit circle, with their singularities located at some of the points θ_i, and of nothing else. In particular, the derivative $f_{(n)}^{(n+1)\prime}(\theta)$ is identically zero within all the open intervals defining the sections. However, this derivative cannot be identically zero over the whole unit circle.

In fact, it is impossible to have a non-zero piecewise polynomial real function of order n, such as the one described above, that is also continuous and differentiable to the order n on the whole unit circle. It follows that $f_{(n)}(\theta)$ can be globally differentiable at most to order $n-1$, so that its $n^{\rm th}$ derivative is a discontinuous function, and therefore its $(n+1)^{\rm th}$ derivative already gives rise to singular distributions.

In short, every real function that is piecewise polynomial on the unit circle, of order n, when differentiated $n+1$ times, so that it becomes zero within the open intervals corresponding to the existing sections, will always result in the superposition of some non-empty set of singular distributions with their singularities located at the points between two consecutive sections. Furthermore, *only* functions of this form give rise to such superpositions of singular distributions and of nothing else.

2.3 Locally Integrable Real Functions

In order to deal with the class of non-integrable real functions we are to discuss here, we must now introduce a different concept of integrability, which is known as *local integrability.* For integrable real functions in compact domains this is equivalent to both plain integrability and absolute integrability, since all real functions discussed in this paper are assumed to be Lebesgue-measurable functions. Here is the definition, formulated in a way appropriate for our case here.

Definition 1: *A real function $f(\theta)$ is locally integrable on the unit circle if it is integrable on every closed interval contained within that domain.*

In this case we say that the real function is *locally integrable everywhere* on the unit circle. This definition selects the same set of real functions as plain integrability, and also the same set of real functions as absolute integrability, given that all real functions discussed in this paper are assumed to be Lebesgue-measurable functions. This concept of local integrability can now be generalized to functions that have a finite number of non-integrable singularities, giving rise to the concept of *local integrability almost everywhere.* Given a real function $f(\theta)$ defined on the unit circle, which has a finite number $N \geq 1$ of non-integrable singularities at isolated points on the unit circle, corresponding to the angles $\{\theta_1, \ldots, \theta_N\}$, we introduce the definition that follows.

Definition 2: *A real function $f(\theta)$ is locally integrable almost everywhere on the unit circle if it is integrable on every closed interval contained within that domain, that does not contain any of the points where the function has non-integrable singularities, of which there is a finite number.*

This definition begins to characterize the class of non-integrable real functions whose relationship with inner analytic functions we will consider here. Note that we include in this definition the fact that the number of non-integrable singularities must be finite.

2.4 Extended Fourier Theory

In [5] we showed that the whole Fourier theory of integrable real functions is contained within the complex-analytic structure presented in [1]. There we also established an extension of the Fourier theory to essentially all inner analytic functions. In particular, we introduced the concept of an exponentially bounded sequence of either Taylor or Fourier coefficients. Here is that definition: given an arbitrary ordered set of complex coefficients a_k, for $k \in \{0, 1, 2, 3, \ldots, \infty\}$, if they satisfy the condition that

$$\lim_{k\to\infty} |a_k|\,\mathrm{e}^{-Ck} = 0, \tag{7}$$

for all real $C > 0$, then we say that the sequence of coefficients a_k is *exponentially bounded*. This definition applies equally well to either complex Taylor coefficients or to real Fourier coefficients. In the construction of the inner analytic functions presented in [1], if α_0, α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, are the Fourier coefficients of an integrable real function $f(\theta)$, then we construct from them the set of complex Taylor coefficients $c_0 = \alpha_0/2$ and $c_k = \alpha_k - \imath\beta_k$, for $k \in \{1, 2, 3, \ldots, \infty\}$. We also showed in [5] that the condition that the sequence of Taylor coefficients c_k is exponentially bounded is equivalent to the condition that the corresponding sequences of Fourier coefficients α_k and β_k are both exponentially bounded. Another result that was obtained in [5] is that the condition that the sequence of Taylor coefficients c_k is exponentially bounded is sufficient to ensure that the corresponding power series is convergent within the open unit disk, and therefore converges to an inner analytic function. Finally, we showed in that paper that the condition that the Fourier coefficients are exponentially bounded suffices to guarantee that the *regulated* Fourier series given by

$$u(\rho, \theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \rho^k \left[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)\right] \tag{8}$$

converges absolutely and uniformly for $0 < \rho < 1$, and that $f(\theta)$ can be obtained as the $\rho \to 1_{(-)}$ limit of this regulated Fourier series almost everywhere on the unit circle,

$$f(\theta) = \frac{\alpha_0}{2} + \lim_{\rho\to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)\right]. \tag{9}$$

This has the effect of extending the applicability of the Fourier theory far beyond the realm of integrable real functions, including, for example, the singular Schwartz distributions.

3 General Statement of the Problem

In [1] we showed that every integrable real function can be related to a unique inner analytic function that corresponds to it. Since all the inner analytic functions can be organized in integral-differential chains, it follows that each integrable real function can be assigned a unique place in a unique integral-differential chain. However, note that all such functions are assigned only to a part of each one of these chains, the part where the singularities of the inner analytic functions are typically either soft or borderline hard. Still, the complete integral-differential chains do exist and, in particular, do extend indefinitely in the differentiation direction. As we will see, in general non-integrable real functions can be related to inner analytic functions that have one or more hard singularities which are not borderline hard ones, being therefore on the differentiation side of the integral-differential chains.

In addition to this, in [2] we showed that integrable real functions that are piecewise polynomial, and only those, belong to integral-differential chains that have only superpositions of one or more delta "functions" and derivatives of delta "functions" on their differentiation side, if one goes in the differentiation direction far enough along their integral-differential chains. Therefore, given any integrable real function which is *not* a piecewise polynomial function, and is such that the corresponding inner analytic function has one or more singularities on the unit circle, a type of real function of which there certainly are many, the integral-differential chain to which it belongs contains, if we travel far enough in the differentiation direction along the chain, real functions which are *not* integrable at one or more of their singular points.

Starting from an integrable real function $f(\theta)$ with Fourier coefficients α_k and β_k, leading to the complex Taylor coefficients c_k of an inner analytic function $w(z)$, thus following the construction presented in [1], it is not difficult to see from the definition of the angular derivative that after n angular differentiations of $w(z)$ we get a proper inner analytic function $w^{n\bullet}(z)$ with coefficients given by $c_k^{(n)} = \imath^n k^n c_k$. Starting from the convergent Taylor series of $w(z)$ we have for its $n^{\rm th}$ angular derivative

$$\begin{aligned} w(z) &= \sum_{k=0}^{\infty} c_k z^k \quad \Rightarrow \\ w^{n\bullet}(z) &= \sum_{k=0}^{\infty} c_k^{(n)} z^k \\ &= \imath^n \sum_{k=0}^{\infty} (k^n c_k)\, z^k, \end{aligned} \tag{10}$$

for $n \in \{0, 1, 2, 3, \ldots, \infty\}$. As we showed in [1], the coefficients c_k associated to an integrable real function $f(\theta)$ are limited as functions of k. For integrable functions $f(\theta)$ which are not analytic on the whole unit circle the coefficients c_k typically go to zero when $k \to \infty$, at a pace which is slower than exponentially fast. We thus see that non-integrable real functions obtained as derivatives of integrable real functions are associated to Taylor coefficients which typically diverge to infinity as a positive power of k in the $k \to \infty$ limit.

Therefore, we must conclude that there is a large class of non-integrable real functions which are still related to corresponding inner analytic functions such as $w^{n\bullet}(z)$, and that can be obtained as the $\rho \to 1_{(-)}$ limits to the unit circle of the real parts of these inner analytic functions. This is the case so long as the $\rho \to 1_{(-)}$ limits of these inner analytic functions exist almost everywhere on the unit circle. Since angular differentiation conserves the set of singular points of $w(z)$, at all points where $w(z)$ is analytic, so is $w^{n\bullet}(z)$, and therefore the limit exists there. Where $w(z)$ has soft singularities with a large enough degree of softness, namely larger that or equal to $n+1$, the angular derivative $w^{n\bullet}(z)$ will also have soft singularities, and again the limit exists there. Since the $\rho \to 1_{(-)}$ limits do not exist at hard singularities, and in analogy to what we discussed in [1] in the case of the borderline hard singularities of integrable real functions, we see that we must impose some limitations on the non-integrable real functions to be considered here. Specifically, we assume that the numbers of non-integrable singularities of these functions are finite.

By the argument above we can immediately determine the existence of many non-integrable real functions which are related to given inner analytic functions. Given any inner analytic function which is not simply a superposition of inner analytic functions associated to singular distributions, and which has one or more singularities on the unit circle, by differentiating it a sufficient number of times we produce an inner analytic function which corresponds to a non-integrable real function. However, given only a non-integrable real function, the problem of the determination of the corresponding inner analytic function is not so immediate. In fact, the problem stated in this way does not really have a unique solution, because such a real function is not really defined on the whole periodic interval, but only on a strict subset of it. The definition of a non-integrable real function must leave out the set of points on the unit circle where it has non-integrable singularities, and thus diverges to infinity. Therefore, one can superpose to such a real function any set of delta "functions" and derivatives of delta "functions", that have their singularities located at those same singular points, without changing at all the definition of the original non-integrable real function. It then follows that we can add to any inner analytic function that corresponds to the non-integrable real function linear combinations of the inner analytic functions that correspond to the delta "function" and to the derivatives of the delta "function", which were given explicitly in [2], chosen so that their singularities are located at the points where the non-integrable real function is not defined, without changing the correspondence between the non-integrable real function and the respective inner analytic function.

Let us state, then, the problem we propose to investigate in this paper, relating to non-integrable real functions and corresponding inner analytic functions. We want to find out how to determine an inner analytic function, given only the non-integrable real function, that reproduces that real function almost everywhere in its domain of definition as the $\rho \to 1_{(-)}$ limit of its real part. We will give here a solution to this problem for a rather large class of non-integrable real functions, on which we will impose, however, some limitations. The first limitation will be, of course, that the real function $f(\theta)$ be locally integrable almost everywhere. This means, in particular, that its number of hard singular points, either integrable or non-integrable, must be finite. The other condition is that the non-integrable singular points of $f(\theta)$ have finite degrees of hardness, as defined and discussed for inner analytic functions in [1], and translated to real functions in this paper, at the beginning of Section 2.

4 Representation of Non-Integrable Real Functions

First of all, let us describe, in general lines, the algorithm we propose to use in order to determine an inner analytic function that corresponds to a given non-integrable real function. Given a non-integrable real function $f(\theta)$, which is however locally integrable almost everywhere, and whose hard singularities have a finite maximum degree of hardness n, we sectionally integrate it n times. Since by hypothesis all the non-integrable singularities of $f(\theta)$ have degrees of hardness of at most n, the resulting function $f^{-n\prime}(\theta)$ is in fact an integrable one on the whole unit circle. Therefore, we may use it to construct the corresponding inner analytic function, which we will name $w^{-n\bullet}(z)$, using the construction presented in [1]. Having this inner analytic function, we then calculate its $n^{\rm th}$ angular derivative, in order to obtain $w(z)$, which is the inner analytic function that corresponds to the non-integrable real function $f(\theta)$. However, the n-fold angular differentiation process produces only the *proper* inner analytic function $w_p(z)$ associated to $w(z)$, and therefore produces $f(\theta)$ only up to an overall constant related to the whole unit circle, as we will soon see. Of course this scheme will succeed if and only if the non-integrable singular points of $f(\theta)$ all have finite degrees of hardness, with a maximum value of n.

In this section we will prove the following theorem.

Theorem 1: *Every non-integrable real function defined almost everywhere on the periodic interval, which is locally integrable almost everywhere, and which is such that its non-integrable singularities have finite degrees of hardness, can be represented by an inner analytic function, and can be recovered almost everywhere on its domain of definition by means of the limit to the unit circle of the real part of that inner analytic function.*

Before we attempt to prove the theorem, let us establish some notation for a sectionally defined real function $f(\theta)$, that will be similar to the one adopted for the piecewise polynomial functions discussed in [2], which is also given in Equation (6). Since a non-integrable real function $f(\theta)$ which is locally integrable almost everywhere is not defined at the singular points corresponding to the angles θ_i, for $i \in \{1, \ldots, N\}$, it is in fact only sectionally defined, in N open intervals between consecutive singularities, as given in Equation (5). Let us therefore specify the definition of $f(\theta)$ as a set of $N \geq 1$ functions $f_i(\theta)$ defined on the N sections (θ_i, θ_{i+1}),

$$f(\theta) = \left\{ f_i(\theta), i \in \{1, \ldots, N\} \right\}, \tag{11}$$

where, as before, we adopt the convention that every section (θ_i, θ_{i+1}) is numbered after the singular point θ_i at its left end.

As a preliminary to the proof of the theorem, some considerations are in order, regarding the multiple sectional integration of such non-integrable functions. Note that, if $f(\theta)$ is locally integrable almost everywhere, then it is integrable within each one of the N open intervals that define the sections. More precisely, it is integrable on every closed interval contained within one of these open intervals. In terms of the sectional functions, since $f_i(\theta)$ is integrable within its section, we may define a piecewise primitive for $f(\theta)$, by simply integrating each function $f_i(\theta)$ within the corresponding open interval, starting at some arbitrary reference point $\theta_{0,i}$ strictly within that open interval. This will define the piecewise primitive

$$\begin{aligned} f^{-1\prime}(\theta) &= \left\{ f_i^{-1\prime}(\theta), i \in \{1, \ldots, N\} \right\}, \\ f_i^{-1\prime}(\theta) &= \int_{\theta_{0,i}}^{\theta} d\theta' \, f_i(\theta'), \end{aligned} \tag{12}$$

where we have that $\theta_i < \theta < \theta_{i+1}$ and that $\theta_i < \theta_{0,i} < \theta_{i+1}$. Note that, since $f_i(\theta)$ is integrable on every closed interval contained within its section, it follows that $f_i^{-1\prime}(\theta)$ is limited on these closed intervals, and therefore is also integrable on them. Therefore, this process of sectional integration of the sectional functions can be iterated indefinitely. If we iterate this process of sectional integration, we obtain further piecewise primitives $f^{-2\prime}(\theta)$, $f^{-3\prime}(\theta)$, and so on. Note also that, during this process of multiple sectional integration, some non-integrable singularities, having a lower degree of hardness, may become integrable before the others. In this case we might ignore the singularities which became integrable, from that point on in the multiple integration process, which therefore effectively reduces the number of sections, but for simplicity we choose not to do that, and thus to keep the set of sections constant. Note that, in any case, we do continue the process of sectional integration until *all* the singularities have become integrable, of course.

Although for definiteness we are integrating from some particular reference points $\theta_{0,i}$ within each section, our objective here is actually to construct primitives, and therefore we may ignore the particular values chosen for $\theta_{0,i}$ if at each step in this iterative process we add an arbitrary integration constant to the primitive in each section, so that after n such successive integrations a polynomial of degree $n-1$, with n arbitrary coefficients, will have been added to the n^{th} primitive in the i^{th} section. We will express this as follows,

$$\begin{aligned} f^{-n\prime}(\theta) &= \left\{ f_i^{-n\prime}(\theta) + P_i^{(n-1)}(\theta), i \in \{1, \ldots, N\} \right\} \\ &= \left\{ f_i^{-n\prime}(\theta), i \in \{1, \ldots, N\} \right\} + P_{(n-1)}(\theta), \\ P_{(n-1)}(\theta) &= \left\{ P_i^{(n-1)}(\theta), i \in \{1, \ldots, N\} \right\}, \end{aligned} \tag{13}$$

where $f^{-n\prime}(\theta)$ is the most general piecewise $n^{\rm th}$ primitive of $f(\theta)$, $f_i^{-n\prime}(\theta)$ is an arbitrary $n^{\rm th}$ primitive of $f_i(\theta)$ in the $i^{\rm th}$ section, $P_i^{(n-1)}(\theta)$ is an arbitrary polynomial of order $n-1$ in the $i^{\rm th}$ section, and $P_{(n-1)}(\theta)$ is a piecewise polynomial function of order $n-1$, containing therefore n arbitrary constants in each section. Note that $P_{(n-1)}(\theta)$ is always an integrable real function. Note also that, upon subsequent n-fold differentiation of $f^{-n\prime}(\theta)$ with respect to θ, all the arbitrary constants that were added during the multiple integration process are then eliminated, the arbitrary polynomials vanish from the result, and we thus get back the original function $f(\theta)$, within the open intervals that constitute the sections,

Finally, we must emphasize some facts about the behavior of the correspondence between inner analytic functions an integrable real functions under the respective operations of differentiation and integration, that take us along the corresponding integral-differential chain. First, let us recall that, as we saw in [1], both angular differentiation and angular integration produce only *proper* inner analytic functions, and thus always result in null Taylor coefficients c_0, and thus in null Fourier coefficients α_0. Therefore, when we use angular integration and differentiation in our algorithm, we lose all information about these two $k=0$ coefficients. Second, as we also saw in [1], the integration on θ implies that the resulting Fourier coefficient α_0 becomes indeterminate due to the presence of an arbitrary integration constant. It is due to this that we must add arbitrary constants during the process of multiple sectional integration, thus generating the arbitrary piecewise polynomial real function $P_{(n-1)}(\theta)$.

At this point, let us review the algorithm we are to use here. First, starting from the non-integrable real function $f(\theta)$ we go n steps along the integration direction of the integral-differential chain, using sectional integration on θ on the unit circle. This produces the n-fold piecewise primitive $f^{-n\prime}(\theta)$ containing the arbitrary piecewise polynomial real function $P_{(n-1)}(\theta)$. From the globally integrable real function $f^{-n\prime}(\theta)$ we then construct the inner analytic function $w^{-n\cdot}(z)$, using the construction presented in [1]. We then come back in the differentiation direction of the integral-differential chain the same number of steps, using this time angular differentiation of the inner analytic functions within the open unit disk. This produces an inner analytic function associated to $f(\theta)$, except for its coefficient c_0, which means that we recover only a proper inner analytic function, given that it is the result of a series of angular differentiations, and we will therefore denote this function by $w_p(z) = u_p(\rho,\theta) + \imath v_p(\rho,\theta)$. Since this is equivalent to differentiation with respect to θ of the piecewise polynomial real function $P_{(n-1)}(\theta)$, it completely eliminates this function within the open intervals that constitute the sections. However, it also produces the superposition of a set of delta "functions" and derivatives of delta "functions" with support on the singular points between successive sections.

Thus we must conclude that this algorithm necessarily results in an indeterminate Taylor coefficient c_0 of the inner analytic function $w(z)$ which corresponds to the real function $f(\theta)$, so that we recover only the corresponding proper inner analytic function $w_p(z)$, and therefore it results in an equally indeterminate Fourier coefficients α_0 related to the real function $f(\theta)$. Se see, therefore, that in this paper we are compelled to relax, to some extent, and only during the process of construction of the inner analytic functions, the correspondence between the real functions and these inner analytic functions, considering only proper inner analytic functions, and accepting the fact that they will correspond to the non-integrable real functions only up to a global constant over the whole unit circle. Once the proper inner analytic function $w_p(z)$ corresponding to a non-integrable real function $f(\theta)$ has been determined, the relation will be written as

$$f(\theta) \longleftrightarrow \frac{\alpha_0}{2} + w_p(z), \qquad (14)$$

where α_0 is a real constant, which can be determined afterwards, by the simple comparison of the known value of the real part $u_p(1,\theta)$ of $w_p(z)$ and the known value of $f(\theta)$, at any point θ on the unit circle where they do not have hard singularities. If θ_0 is such a point, then we have that $\alpha_0 = 2[f(\theta_0) - u_p(1,\theta_0)]$. Once the coefficient α_0 is thus determined, this finally determines completely the inner analytic function $w(z)$ that corresponds to $f(\theta)$,

$$w(z) = \frac{\alpha_0}{2} + w_p(z). \tag{15}$$

We are now ready to prove the theorem.

Proof 1.1:

In order to prove the theorem, our first task is to show that we can obtain an inner analytic function $w(z)$ from the non-integrable real function $f(\theta)$, which is assumed to be locally integrable almost everywhere. Consider that we execute the iterative piecewise integration process described before n times on $f(\theta)$, where n is the maximum among all the degrees of hardness of the non-integrable hard singularities involved. By doing this we generate a real function $f^{-n\prime}(\theta)$ which has only soft or at most borderline hard singularities on the unit circle, and which is therefore integrable on the whole unit circle.

It follows therefore that we may determine its set of Fourier coefficients, as was done in [1], which we will name $\alpha_0^{(-n)}$, $\alpha_k^{(-n)}$ and $\beta_k^{(-n)}$, for $k \in \{1, 2, 3, \ldots, \infty\}$. From this set of Fourier coefficients we may then define, again as was done in [1], the complex Taylor coefficients $c_k^{(-n)}$, for $k \in \{0, 1, 2, 3, \ldots, \infty\}$. From these coefficients we may then determine the unique inner analytic function that corresponds to the $n^{\rm th}$ piecewise primitive $f^{-n\prime}(\theta)$, which we will name $w^{-n\bullet}(z) = u^{-n\prime}(\rho,\theta) + \imath v^{-n\prime}(\rho,\theta)$. Note that we may use this notation unequivocally because every proper inner analytic function belongs to an infinite integral-differential chain, extending indefinitely to either side, so that we know that there exists in fact a proper inner analytic function $w_p(z)$ associated to

$$w_p^{-n\bullet}(z) = w^{-n\bullet}(z) - c_0^{(-n)}, \tag{16}$$

namely its $n^{\rm th}$ angular derivative. As we have shown in [1], from the $\rho \to 1_{(-)}$ limit to the unit circle of the real part $u^{-n\prime}(\rho,\theta)$ of the inner analytic function $w^{-n\bullet}(z)$ we can recover the integrable real function $f^{-n\prime}(\theta)$, almost everywhere in its domain of definition. We thus have the correspondence for the integrable real function

$$f^{-n\prime}(\theta) \longleftrightarrow w^{-n\bullet}(z). \tag{17}$$

Having done this, we now take the $n^{\rm th}$ angular derivative of the inner analytic function $w^{-n\bullet}(z)$, thus obtaining a proper inner analytic function $w_p(z) = u_p(\rho,\theta) + \imath v_p(\rho,\theta)$. Since n angular differentiations of $w^{-n\bullet}(z)$ correspond to n differentiations with respect to θ of $f^{-n\prime}(\theta)$, and thus completely eliminates the piecewise polynomial real function $P_{(n-1)}(\theta)$ within the domain of definition of $f(\theta)$, it follows that the function $w_p(z)$ is a proper inner analytic function corresponding to the non-integrable real function $f(\theta)$. As discussed before, this correspondence is valid only up to a global real constant yet to be determined, so that we may now write

$$f(\theta) \longrightarrow \frac{\alpha_0}{2} + w_p(z), \tag{18}$$

where α_0 can then be determined as discussed before, by the comparison between the real part $u_p(1,\theta)$ of $w_p(z)$ and $f(\theta)$ at some particular point on the unit circle where they do not have hard singularities. Having determined α_0, and therefore $c_0 = \alpha_0/2$, we may now define the inner analytic function that corresponds to $f(\theta)$,

$$w(z) = c_0 + w_p(z), \tag{19}$$

so that we have the relation leading from $f(\theta)$ to $w(z)$

$$f(\theta) \longrightarrow w(z). \tag{20}$$

This completes the first part of the proof of Theorem 1.

Proof 1.2:

We must now prove that we can recover $f(\theta)$ from the real part $u(\rho, \theta)$ of $w(z)$ in the $\rho \to 1_{(-)}$ limit. In order to do this, we start from the fact that from [1] we know this to be true for the n-fold primitives

$$w^{-n\bullet}(z) \longleftrightarrow f^{-n\prime}(\theta). \tag{21}$$

While the inner analytic function $w^{-n\bullet}(z)$ is given by the power series

$$w^{-n\bullet}(z) = \sum_{k=0}^{\infty} c_k^{(-n)} z^k, \tag{22}$$

as we have shown in [5] the real function $f^{-n\prime}(\theta)$ can be expressed almost everywhere as an regulated Fourier series, even if the Fourier series itself diverges almost everywhere,

$$f^{-n\prime}(\theta) = \frac{\alpha_0^{(-n)}}{2} + \lim_{\rho \to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[\alpha_k^{(-n)} \cos(k\theta) + \beta_k^{(-n)} \sin(k\theta) \right], \tag{23}$$

where we have that $c_0^{(-n)} = \alpha_0^{(-n)}/2$ and that $c_k^{(-n)} = \alpha_k^{(-n)} - \imath \beta_k^{(-n)}$ for $k \in \{1, 2, 3, \ldots, \infty\}$. As we established in [5], since the sequences of Fourier coefficients $\alpha_k^{(-n)}$ and $\beta_k^{(-n)}$ are exponentially bounded, this series is absolutely and uniformly convergent for $0 < \rho < 1$. As we saw in [1], the fact that we can recover $f^{-n\prime}(\theta)$ as the $\rho \to 1_{(-)}$ limit of the real part $u^{-n\prime}(\rho, \theta)$ of $w^{-n\bullet}(z)$ is a consequence of the fact that the two real functions $u^{-n\prime}(1, \theta)$ and $f^{-n\prime}(\theta)$ have exactly the same set of Fourier coefficients. This, is turn, can be expressed as the relations between the Taylor coefficients $c_k^{(-n)}$ associated to $u(\rho, \theta)$ and the Fourier coefficients $\alpha_k^{(-n)}$ and $\beta_k^{(-n)}$ associated to $f(\theta)$, which have just been given above.

Let us now prove that the correspondence between $u^{-n\prime}(\rho, \theta)$ and $f^{-n\prime}(\theta)$ implies the same correspondence between $u^{(-n+1)\prime}(\rho, \theta)$ and $f^{(-n+1)\prime}(\theta)$, when we differentiate the two functions. We can do this by just showing that the relations between the Taylor coefficients and the Fourier coefficients are preserved by this process of differentiation. If we just differentiate the inner analytic function using angular differentiation we get

$$\begin{aligned} w^{(-n+1)\bullet}(z) &= \sum_{k=1}^{\infty} \imath k \, c_k^{(-n)} z^k \\ &= \sum_{k=1}^{\infty} c_k^{(-n+1)} z^k, \end{aligned} \tag{24}$$

from which we have for the Taylor coefficients $c_k^{(-n+1)} = \imath k\, c_k^{(-n)}$, for $k \in \{1, 2, 3, \ldots, \infty\}$. Note that, since this is a convergent power series, we can always differentiate it term-by-term. If we now differentiate the real function using simple differentiation with respect to θ we get

$$\begin{aligned} f^{(-n+1)\prime}(\theta) &= \lim_{\rho\to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[k\, \beta_k^{(-n)} \cos(k\theta) - k\, \alpha_k^{(-n)} \sin(k\theta)\right], \\ &= \lim_{\rho\to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[\alpha_k^{(-n+1)} \cos(k\theta) + \beta_k^{(-n+1)} \sin(k\theta)\right], \end{aligned} \tag{25}$$

from which we have for the corresponding Fourier coefficients that $\alpha_k^{(-n+1)} = k\, \beta_k^{(-n)}$ and also that $\beta_k^{(-n+1)} = -k\, \alpha_k^{(-n)}$, for $k \in \{1, 2, 3, \ldots, \infty\}$. Note that we have, in either case, that $c_0^{(-n+1)} = 0$ and that $\alpha_0^{(-n+1)} = 0$, so that the relation between $c_0^{(-n)}$ and $\alpha_0^{(-n)}$ is in fact preserved. Note also that, since this trigonometric series is uniformly convergent, and in fact is the real part of a convergent complex power series, we may differentiate it term-by-term so long as the series thus obtained is also convergent. Since the Fourier coefficients $\alpha_0^{(-n+1)}$ and $\beta_0^{(-n+1)}$, which increase at most as a power of k when $k \to \infty$, are thus seen to be exponentially bounded, this implies that the series thus obtained is also absolutely and uniformly convergent, as we have shown in [5]. Therefore, we are justified in differentiating the original series term-by-term. We therefore have the relation between the Taylor coefficients $c_k^{(-n+1)}$ and the Fourier coefficients $\alpha_k^{(-n+1)}$ and $\beta_k^{(-n+1)}$,

$$\begin{aligned} c_k^{(-n+1)} &= \imath k\, c_k^{(-n)} \\ &= \imath k \left[\alpha_k^{(-n)} - \imath \beta_k^{(-n)}\right] \\ &= \imath k \left[-\frac{1}{k}\, \beta_k^{(-n+1)} - \imath\, \frac{1}{k}\, \alpha_k^{(-n+1)}\right] \\ &= \alpha_k^{(-n+1)} - \imath \beta_k^{(-n+1)}. \end{aligned} \tag{26}$$

We see therefore that we indeed have that the relation between $c_k^{(-n)}$, $\alpha_k^{(-n)}$ and $\beta_k^{(-n)}$, for $k \in \{1, 2, 3, \ldots, \infty\}$, is also preserved,

$$c_k^{(-n+1)} = \alpha_k^{(-n+1)} - \imath \beta_k^{(-n+1)}, \tag{27}$$

which thus establishes the correspondence for the first derivatives. We may now extend this argument to subsequent derivatives, from $w^{(-n+1)\bullet}(z)$ and $f^{(-n+1)\prime}(\theta)$ all the way to $w(z)$ and $f(\theta)$, by finite induction. Therefore, let us assume the result for the case $(-n + i)$ and show that this implies that it is also valid for the case $(-n + i + 1)$. We assume therefore that we have

$$c_k^{(-n+i)} = \alpha_k^{(-n+i)} - \imath \beta_k^{(-n+i)}, \tag{28}$$

for some positive value of i, where the functions are expressed as the corresponding series

$$\begin{aligned} w^{(-n+i)\bullet}(z) &= \sum_{k=1}^{\infty} c_k^{(-n+i)} z^k, \\ f^{(-n+i)\prime}(\theta) &= \lim_{\rho\to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[\alpha_k^{(-n+i)} \cos(k\theta) + \beta_k^{(-n+i)} \sin(k\theta)\right]. \end{aligned} \tag{29}$$

We may now differentiate either series term-by-term, which we may do for the same reasons as before, thus obtaining

$$\begin{aligned} w^{(-n+i+1)\prime}(z) &= \sum_{k=1}^{\infty} \imath k\, c_k^{(-n+i)} z^k, \\ f^{(-n+i+1)\prime}(\theta) &= \lim_{\rho\to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[k\,\beta_k^{(-n+i)} \cos(k\theta) - k\,\alpha_k^{(-n+i)} \sin(k\theta) \right], \end{aligned} \tag{30}$$

so that we have for the coefficients for the case $(-n+i+1)$

$$\begin{aligned} c_k^{(-n+i+1)} &= \imath k\, c_k^{(-n+i)}, \\ \alpha_k^{(-n+i+1)} &= k\,\beta_k^{(-n+i)}, \\ \beta_k^{(-n+i+1)} &= -k\,\alpha_k^{(-n+i)}, \end{aligned} \tag{31}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. Using now the relations between the coefficients for the case $(-n+i)$ we have

$$\begin{aligned} c_k^{(-n+i+1)} &= \imath k\, c_k^{(-n+i)} \\ &= \imath k \left[\alpha_k^{(-n+i)} - \imath \beta_k^{(-n+i)} \right] \\ &= \imath k \left[-\frac{1}{k}\,\beta_k^{(-n+i+1)} - \imath\,\frac{1}{k}\,\alpha_k^{(-n+i+1)} \right] \\ &= \alpha_k^{(-n+i+1)} - \imath \beta_k^{(-n+i+1)}, \end{aligned} \tag{32}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$, thus showing that the relation between the coefficients is in fact preserved, where we recall that the $k = 0$ coefficients are always zero during this process. This is therefore true for all possible multiple derivatives, all the way to infinity, and in particular it is true for $i = n$, that is, for the coefficients of $w(z)$ and $f(\theta)$. In order to complete the proof in this case all we have to do is to consider the real function

$$g(\theta) = u(1, \theta) - f(\theta), \tag{33}$$

where

$$u(1, \theta) = \lim_{\rho\to 1_{(-)}} u(\rho, \theta). \tag{34}$$

Since the expression of the Fourier coefficients is linear on the functions, and since $u(1, \theta)$ and $f(\theta)$ have exactly the same set of Fourier coefficients, it is clear that all the Fourier coefficients of $g(\theta)$ are zero. Therefore, for the real function $g(\theta)$ we have that $c_k = 0$ for all k, and thus the inner analytic function that corresponds to $g(\theta)$ is the identically null complex function $w_\gamma(z) \equiv 0$. This is an inner analytic function which is, in fact, analytic over the whole complex plane, and which, in particular, is zero over the unit circle, so that we have $g(\theta) \equiv 0$, since the identically zero real function is the *only* integrable real function without removable singularities that corresponds to the identically zero inner analytic function, due to the completeness of the Fourier basis, as was shown in [5]. Note, in particular, that the $\rho \to 1_{(-)}$ limits exist at *all* points of the unit circle in the case of the inner analytic function associated to $g(\theta)$. Since both $u(1, \theta)$ and $f(\theta)$ have non-integrable hard singularities at isolated points on the unit circle, we can conclude only that

$$f(\theta) = \lim_{\rho \to 1_{(-)}} u(\rho, \theta) \tag{35}$$

almost everywhere on the unit circle, or everywhere in the domain of definition of $f(\theta)$, which therefore excludes all the points where the function has hard singularities. Note that the domain of definition of $u(1,\theta)$ is the same as that of $f(\theta)$, because any hard singularities that may have been softened in the process of iterative integration, during the construction of $w(z)$, will have been hardened again in the corresponding process of iterative differentiation. We have therefore the complete correspondence

$$f(\theta) \longleftrightarrow w(z). \tag{36}$$

The inner analytic function $w(z)$ represents the non-integrable real function $f(\theta)$ exactly in the same way as that which was established for integrable real functions in [1]. Note that this proof establishes that the correspondence between the inner analytic functions and the real functions is also valid for all the intermediate cases, from $w^{-n^{\bullet}}(z)$ and $f^{-n\prime}(\theta)$ to $w(z)$ and $f(\theta)$. This completes the second part of the proof of Theorem 1.

Let us once more draw attention to the fact that the inner analytic function $w(z)$ produced in the way described above, while it does correspond to the non-integrable real function $f(\theta)$, is *not* unique. One can add to it any linear combination of inner analytic functions that correspond to singular distributions with their singularities at the singular points on the unit circle corresponding to the angles $\{\theta_1, \ldots, \theta_N\}$, without changing the fact that the non-integrable real function $f(\theta)$ is still recovered as the $\rho \to 1_{(-)}$ limit of the real part of the new inner analytic function obtained in this way, at all points where it is well defined.

Given the inner analytic function $w(z)$ that was obtained from $f(\theta)$ by the process described above, one can then consider defining from it a *reduced* inner analytic function $w_r(z)$ that represents the non-integrable real function in a unique way, without the superposition of any singular distributions. If the N singular points of $w(z)$ are examined in order to determine the existence there of singular distributions, and since each one of these singular distributions is represented by a known unique inner analytic function of its own, as was shown in [2], one can simply subtract from $w(z)$ the appropriate linear combination of inner analytic functions of the singular distributions present, in order to obtain an inner analytic function that represents the non-integrable real function $f(\theta)$ alone, without any singular distributions superposed to it.

The simplest way to do this is to examine the result of every angular differentiation during the process leading from $w^{-n^{\bullet}}(z)$ to $w(z)$. At each step one can verify whether or not the angular differentiation has generated one or more Dirac delta "functions" at some of the singular points. This is a simple thing to verify, because the occurrence of a delta "function" at a certain point is always preceded by the occurrence of a finite discontinuity of the real function at that point. This can also be done by the determination of the type and orientation of the singularities at these points, since the Dirac delta "functions" are associated to inner analytic functions with simple poles that have a specific orientation with respect to the unit circle. One can then subtract from the proper inner analytic function obtained at that point in the iterative differentiation process the inner analytic functions corresponding to the delta "functions" at the appropriate points. By doing this one guarantees that no derivatives of delta "functions" will ever arise during the process of multiple differentiation. After one determines α_0 at the last step of the process, this will lead to a reduced inner analytic function $w_r(z)$ which includes no singular distributions at all, and that hence corresponds to $f(\theta)$ in a unique and simple way,

$$f(\theta) \longleftrightarrow w_r(z), \tag{37}$$

that does not include the superposition of any singular distributions with support on the singular points between successive sections.

5 Representation in the Extended Fourier Theory

Assuming that, given a non-integrable real function $f(\theta)$, the corresponding reduced inner analytic function $w_r(z) = u_r(\rho,\theta) + \imath v_r(\rho,\theta)$ has been determined, we may now consider determining a unique set of Fourier coefficients to be associated to the non-integrable real function $f(\theta)$. Of these, α_0 has already been determined, through the comparison of $u_r(1,\theta)$ and $f(\theta)$ at some point of the unit circle where they do not have hard singularities. It follows that $c_0 = \alpha_0/2$ has also been determined. From the Taylor series of $w_r(z)$,

$$w_r(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{38}$$

we have the values of all the other Taylor coefficients c_k, for $k \in \{1, 2, 3, \ldots, \infty\}$. Given that in this case we have that $c_k = \alpha_k - \imath\beta_k$, we immediately get the values of all the other Fourier coefficients α_k and β_k. Note that this construction has the effect of associating a complete set of Fourier coefficients α_0, α_k and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, to the non-integrable real function $f(\theta)$. In particular, the association of α_0 has the effect of attributing an *average value* to the non-integrable real function $f(\theta)$, which has been defined via an analytic process.

Although these coefficients obviously cannot be written in terms of integrals involving $f(\theta)$ in the usual way, all of them except for α_0 can, in fact, be written as a certain set of integrals. In order to do this, we start by considering the $n^{\rm th}$ angular primitive $w_r^{-n\bullet}(z)$ of the reduced inner analytic function $w_r(z)$, which is, therefore, a proper inner analytic function, and the known associated Fourier and Taylor coefficients, which we will name $\alpha_0^{(-n)}$, $\alpha_k^{(-n)}$ and $\beta_k^{(-n)}$, for $k \in \{1, 2, 3, \ldots, \infty\}$, and $c_k^{(-n)}$, for $k \in \{0, 1, 2, 3, \ldots, \infty\}$. Since it has at most borderline hard singularities on the unit circle, this inner analytic function corresponds to an integrable real function $f_r^{-n\prime}(\theta)$ on that circle,

$$f_r^{-n\prime}(\theta) = \lim_{\rho\to 1_{(-)}} u_r^{-n\bullet}(\rho,\theta), \tag{39}$$

so that the Fourier coefficients of $f_r^{-n\prime}(\theta)$ can be written as integrals involving this function,

$$\begin{aligned} \alpha_k^{(-n)} &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f_r^{-n\prime}(\theta), \\ \beta_k^{(-n)} &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f_r^{-n\prime}(\theta), \end{aligned} \tag{40}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. We now recall that we have for the Taylor series of the $n^{\rm th}$ angular primitive of $w_r(z)$,

$$w_r^{-n\bullet}(z) = \sum_{k=1}^{\infty} c_k^{(-n)} z^k, \tag{41}$$

and therefore the $n^{\rm th}$ angular derivative of this equation is given by

$$w_r^{0\bullet}(z) = \sum_{k=1}^{\infty} \left[\imath^n k^n c_k^{(-n)} \right] z^k, \tag{42}$$

where $w_r^{0\bullet}(z)$ is the proper inner analytic function associated to $w_r(z)$. It thus follows that we have for the Taylor coefficients c_k of $w_r^{0\bullet}(z)$, which are also the Taylor coefficients of $w_r(z)$, for $k \in \{1, 2, 3, \ldots, \infty\}$,

$$c_k = \imath^n k^n c_k^{(-n)}. \tag{43}$$

Since we know that the coefficients $c_k^{(-n)}$, being the Taylor coefficients associated to an integrable real function, are limited for all k, we may now conclude that the coefficients c_k of the non-integrable real function may diverge with k, but not faster than the power k^n. We may write this relation in terms of the Fourier coefficients α_k and β_k associated to the non-integrable real function $f(\theta)$, for $k \in \{1, 2, 3, \ldots, \infty\}$, if we recall from [1] that $c_k = \alpha_k - \imath\beta_k$, and also that $c_k^{(-n)} = \alpha_k^{(-n)} - \imath\beta_k^{(-n)}$,

$$\begin{aligned} \alpha_k - \imath\beta_k &= \imath^n k^n \left[\alpha_k^{(-n)} - \imath\beta_k^{(-n)} \right] \\ &= \imath^n k^n \alpha_k^{(-n)} - \imath^{n+1} k^n \beta_k^{(-n)}. \end{aligned} \tag{44}$$

We now see that the relations between the Fourier coefficients (α_k, β_k) and the Fourier coefficients $\left[\alpha_k^{(-n)}, \beta_k^{(-n)}\right]$ depend on the parity of n. For even $n = 2j$ we have

$$\begin{aligned} \alpha_k &= (-1)^j k^n \alpha_k^{(-n)}, \\ \beta_k &= (-1)^j k^n \beta_k^{(-n)}, \end{aligned} \tag{45}$$

while for odd $n = 2j + 1$ we have

$$\begin{aligned} \alpha_k &= (-1)^j k^n \beta_k^{(-n)}, \\ \beta_k &= (-1)^{j+1} k^n \alpha_k^{(-n)}. \end{aligned} \tag{46}$$

Since we have the coefficients $\alpha_k^{(-n)}$ and $\beta_k^{(-n)}$ written as integrals, we may now write α_k and β_k as integrals, first for the case of even $n = 2j$,

$$\begin{aligned} \alpha_k &= \frac{(-1)^j k^n}{\pi} \int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f_r^{-n\prime}(\theta), \\ \beta_k &= \frac{(-1)^j k^n}{\pi} \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f_r^{-n\prime}(\theta), \end{aligned} \tag{47}$$

and then for the case of odd $n = 2j + 1$,

$$\begin{aligned} \alpha_k &= \frac{(-1)^j k^n}{\pi} \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) f_r^{-n\prime}(\theta), \\ \beta_k &= \frac{(-1)^{j+1} k^n}{\pi} \int_{-\pi}^{\pi} d\theta\, \cos(k\theta) f_r^{-n\prime}(\theta). \end{aligned} \tag{48}$$

Since the integrals shown are all limited as functions of k, given that $f_r^{-n\prime}(\theta)$ is an integrable real function, once again it is apparent that the coefficients α_k and β_k which are associated to $f_r(\theta)$ typically diverge as a positive power of k when $k \to \infty$. Note that, when the real function $f(\theta)$ is integrable on the whole unit circle, we are reduced to the case $n = 0$, so that the expressions for the even case reduce to the usual ones for the Fourier coefficients.

In a previous paper [5] we showed that the whole Fourier theory of integrable real functions is contained in the complex-analytic structure introduced in [1]. We also extended that Fourier theory to include, not only the singular distributions discussed in [2], but essentially the whole space of inner analytic functions. We now observe that the sequences of complex Taylor coefficients c_k in Equation (43), which go to infinity with k not faster than a power, are exponentially bounded, according to the definition of that term given in [5], and repeated in Equation (7). This in itself suffices to show that the corresponding power series converges to an inner analytic function, as was shown in that paper. It also implies, as was also shown there, that the two sequences of real coefficients α_k and β_k associated to $f(\theta)$ are both also exponentially bounded. As a consequence of all this, the non-integrable real functions we are discussing in this paper can be expressed as regulated Fourier series, as given in Equation (8), thus using the summation rule that was presented in [5],

$$f(\theta) = \frac{\alpha_0}{2} + \lim_{\rho \to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k \left[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)\right], \tag{49}$$

which is equivalent to the fact that the non-integrable real functions which we discussed here can be obtained as the $\rho \to 1_{(-)}$ limits of the real parts of inner analytic functions. We see therefore that the class of non-integrable real functions which we are examining here is also contained in the extended Fourier theory presented in [5].

6 Conclusions and Outlook

We have extended the close and deep relationship established in previous papers [1, 2], between, on the one hand, integrable real functions and singular Schwartz distributions, and, in the other hand, complex analytic functions in the unit disk centered at the origin of the complex plane, to include a large class of non-integrable real functions. This close relationship between real functions and related objects, and complex analytic functions, allows one to use the powerful and extremely well-known machinery of complex analysis to deal with the real functions and related objects in a very robust way, even if these objects are very far from being analytic. The concept of integral-differential chains of proper inner analytic functions, which we introduced in [1], played a central role in the analysis presented.

One does not usually associate non-differentiable, discontinuous and unbounded real functions, as well as singular distributions, with single analytic functions. Therefore, it may come as a bit of a surprise that, as was established in [1, 2], essentially *all* integrable real functions, as well as all *all* singular Schwartz distributions, are given by the real parts of certain inner analytic functions on the open unit disk, in the limit in which one approaches the unit circle. This surprise is now further compounded by the fact that inner analytic functions can represent a large class of non-integrable real functions as well.

There are still more inner analytic functions within the open unit disk than those that were examined here and in [1, 2], in relation to integrable real functions, singular distributions and non-integrable real functions. One important limitation in the arguments presented here is that requiring that there be only a finite number of non-integrable hard singularities. It may be possible, perhaps, to lift this limitation, allowing for a denumerably infinite set of such non-integrable singularities. It is probably not possible, however, to allow for a densely distributed set of such singularities. Possibly, the limitation that the number of non-integrable hard singularities be finite may be exchanged for the limitation that the number of *accumulation points* of a denumerably infinite set of singular points with non-integrable hard singularities be finite. We may conjecture that the following condition might work: consider the set of all closed intervals contained in the unit circle on which the function is integrable; consider the point-set infinite union of all such intervals; if the resulting set has measure 2π, then it may be possible to show that the real function can still be represented by an inner analytic function $w(z)$, and thus by a definite set of Fourier coefficients. This condition certainly holds for the cases discussed in this paper, and may even turn out to be the most general possible condition leading to the results.

One interesting aspect of the work presented here is that we obtain a definite set of Fourier coefficients associated to the non-integrable real function $f(\theta)$. It is particularly interesting to note the fact that, when we determine α_0 during the construction of the corresponding inner analytic function $w(z)$, we are in effect *defining*, by analytic means, the average value, over the unit circle, of the non-integrable function $f(\theta)$, for which such a concept was not previously defined at all. A similar remark can be made for all the other Fourier coefficients as well. Just as was the case for integrable real functions and singular Schwartz distributions, the non-integrable real functions examined here can be said to be represented directly by their sequences of Fourier coefficients.

One way to interpret the structure presented here is that, although the non-integrable real function $f(\theta)$ is defined within separate sections, with no predetermined relations among them, it is always possible to define an inner analytic function that reproduces the non-integrable real function correctly, strictly within each one of these sections, and therefore connects them all to one another in an analytic way, just as the interior of the unit disk connects the arcs of the unit circle that correspond to the sections.

We believe that the results presented here enlarge the new perspective for the analysis of real functions which was established in [1]. This development confirms the opinion expressed there that the use of the theory of complex analytic functions makes it a rather deep and powerful point of view. Since complex analysis and analytic functions constitute in fact such a powerful tool, with so many applications in almost all areas of mathematics and physics, it is to be hoped that further applications of the ideas explored here will in due time present themselves.

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