# Complex Analysis of Real Functions

## II: Singular Schwartz Distributions

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**Abstract**

In the context of the complex-analytic structure within the unit disk centered at the origin of the complex plane, that was presented in a previous paper, we show that singular Schwartz distributions can be represented within that same structure, so long as one defines the limits involved in an appropriate way. In that previous paper it was shown that essentially all integrable real functions can be represented within the complex-analytic structure. The infinite collection of singular objects which we analyze here can thus be represented side by side with those real functions, thus allowing all these objects to be treated in a unified way.

## 1 Introduction

In a previous paper [1] we introduced a certain complex-analytic structure within the unit disk of the complex plane, and showed that one can represent essentially all integrable real functions within that structure. In this paper we will show that one can represent within the same structure the singular objects known as distributions, loosely in the sense of the Schwartz theory of distributions [2, 3], which are also known as generalized real functions. All these objects will be interpreted as parts of this larger complex-analytic structure, within which they can be treated and manipulated in a robust and unified way.

In Sections 2 and 3 we will establish the relation between the complex-analytic structure and the singular distributions. There we will show that one obtains these objects through the properties of certain limits to the unit circle, involving a particular set of inner analytic functions, which will be presented explicitly. Following what was shown in [1] for integrable real functions, each singular distribution will be associated to a corresponding inner analytic function. In fact, we will show that the entire space of all singular Schwartz distributions defined within a compact domain is contained within this complex-analytic structure. In Section 4 we will analyze a certain collection of integrable real functions which are closely related to the singular distributions, through the concept of infinite integral-differential chains of functions.

For ease of reference, we include here a one-page synopsis of the complex-analytic structure introduced in [1]. It consists of certain elements within complex analysis [4], as well as of their main properties.

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**Synopsis:** The Complex-Analytic Structure

An *inner analytic function* $w(z)$ is simply a complex function which is analytic within the open unit disk. An inner analytic function that has the additional property that $w(0) = 0$ is a *proper inner analytic function*. The *angular derivative* of an inner analytic function is defined by

$$w^{\bullet}(z) = \imath z \, \frac{dw(z)}{dz}.$$

By construction we have that $w^{\bullet}(0) = 0$, for all $w(z)$. The *angular primitive* of an inner analytic function is defined by

$$w^{-1\bullet}(z) = -\imath \int_{0}^{z} dz' \, \frac{w(z') - w(0)}{z'}.$$

By construction we have that $w^{-1\bullet}(0) = 0$, for all $w(z)$. In terms of a system of polar coordinates $(\rho, \theta)$ on the complex plane, these two analytic operations are equivalent to differentiation and integration with respect to $\theta$, taken at constant $\rho$. These two operations stay within the space of inner analytic functions, they also stay within the space of proper inner analytic functions, and they are the inverses of one another. Using these operations, and starting from any proper inner analytic function $w^{0\bullet}(z)$, one constructs an infinite *integral-differential chain* of proper inner analytic functions,

$$\left\{ \ldots, w^{-3\bullet}(z), w^{-2\bullet}(z), w^{-1\bullet}(z), w^{0\bullet}(z), w^{1\bullet}(z), w^{2\bullet}(z), w^{3\bullet}(z), \ldots \right\}.$$

Two different such integral-differential chains cannot ever intersect each other. There is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, namely the null chain, in which all members are the null function $w(z) \equiv 0$.

A general scheme for the classification of all possible singularities of inner analytic functions is established. A singularity of an inner analytic function $w(z)$ at a point $z_1$ on the unit circle is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity*. Angular integration takes soft singularities to other soft singularities, and angular differentiation takes hard singularities to other hard singularities.

Gradations of softness and hardness are then established. A hard singularity that becomes a soft one by means of a single angular integration is a *borderline hard* singularity, with degree of hardness zero. The *degree of softness* of a soft singularity is the number of angular differentiations that result in a borderline hard singularity, and the *degree of hardness* of a hard singularity is the number of angular integrations that result in a borderline hard singularity. Singularities which are either soft or borderline hard are integrable ones. Hard singularities which are not borderline hard are non-integrable ones.

Given an integrable real function $f(\theta)$ on the unit circle, one can construct from it a unique corresponding inner analytic function $w(z)$. Real functions are obtained through the $\rho \to 1_{(-)}$ limit of the real and imaginary parts of each such inner analytic function and, in particular, the real function $f(\theta)$ is obtained from the real part of $w(z)$ in this limit. The pair of real functions obtained from the real and imaginary parts of one and the same inner analytic function are said to be mutually Fourier-conjugate real functions.

Singularities of real functions can be classified in a way which is analogous to the corresponding complex classification. Integrable real functions are typically associated with inner analytic functions that have singularities which are either soft or at most borderline hard. This ends our synopsis.

When we discuss real functions in this paper, some properties will be globally assumed for these functions, just as was done in [1]. These are rather weak conditions to be imposed on these functions, that will be in force throughout this paper. It is to be understood, without any need for further comment, that these conditions are valid whenever real functions appear in the arguments. These weak conditions certainly hold for any integrable real functions that are obtained as restrictions of corresponding inner analytic functions to the unit circle.

The most basic condition is that the real functions must be measurable in the sense of Lebesgue, with the usual Lebesgue measure [5, 6]. The second global condition we will impose is that the functions have no removable singularities. The third and last global condition is that the number of hard singularities on the unit circle be finite, and hence that they be all isolated from one another. There will be no limitation on the number of soft singularities.

The material contained in this paper is a development, reorganization and extension of some of the material found, sometimes still in rather rudimentary form, in the papers [7–11].

## 2 The Dirac Delta "Function"

This is where we begin the discussion of inner analytic functions that have hard singularities with strictly positive degrees of hardness. Let us start by simply introducing a certain particular inner analytic function of $z$. If $z_1$ is a point on the unit circle, this function is defined as a very simple rational function of $z$,

$$w_\delta(z, z_1) = \frac{1}{2\pi} - \frac{1}{\pi}\frac{z}{z - z_1}. \quad (1)$$

This inner analytic function has a single point of singularity, which is a simple pole at $z_1$. This is a hard singularity with degree of hardness equal to one. Our objective here is to examine the properties of the real part $u_\delta(\rho, \theta, \theta_1)$ of this inner analytic function,

$$w_\delta(z, z_1) = u_\delta(\rho, \theta, \theta_1) + \imath v_\delta(\rho, \theta, \theta_1).$$

We will prove that in the $\rho \to 1_{(-)}$ limit $u_\delta(\rho, \theta, \theta_1)$ can be interpreted as a *Schwartz distribution* [2,3], namely as the singular object known as the *Dirac delta "function"*, which we will denote by $\delta(\theta - \theta_1)$. This object is also known as a *generalized real function*, since it is not really a real function in the usual sense of the term. In the Schwartz theory of distributions this object plays the role of an integration kernel for a certain distribution. Note that $w_\delta(z, z_1)$ can, in fact, be written explicitly as a function of $\rho$ and $\theta - \theta_1$. Since we have that $z = \rho \exp(\imath\theta)$ and that $z_1 = \exp(\imath\theta_1)$, we have at once that

$$w_\delta(z, z_1) = \frac{1}{2\pi} - \frac{1}{\pi}\frac{\rho\, e^{\imath(\theta-\theta_1)}}{\rho\, e^{\imath(\theta-\theta_1)} - 1}.$$

The definition of the Dirac delta "function" is that it is a symbol for a limiting process, which satisfies certain conditions. In our case here the limiting process will be the limit $\rho \to 1_{(-)}$ from the open unit disk to the unit circle. The limit of $u_\delta(\rho, \theta, \theta_1)$ represents the delta "function" in the sense that it satisfies the conditions that follow.

1. The defining limit of $\delta(\theta - \theta_1)$ tends to zero when one takes the $\rho \to 1_{(-)}$ limit while keeping $\theta \neq \theta_1$.

2. The defining limit of $\delta(\theta-\theta_1)$ diverges to positive infinity when one takes the $\rho \to 1_{(-)}$ limit with $\theta = \theta_1$.

3. In the $\rho \to 1_{(-)}$ limit the integral

$$\int_a^b d\theta\, \delta(\theta - \theta_1) = 1$$

has the value shown, for any open interval $(a, b)$ which contains the point $\theta_1$.

4. Given any continuous integrable function $g(\theta)$, in the $\rho \to 1_{(-)}$ limit the integral

$$\int_a^b d\theta\, g(\theta)\delta(\theta - \theta_1) = g(\theta_1)$$

has the value shown, for any open interval $(a, b)$ which contains the point $\theta_1$.

This is the usual form of this condition, when it is formulated in strictly real terms. However, we will impose a slight additional restriction on the real functions $g(\theta)$, by assuming that the limit to the point $z_1$ on the unit circle that corresponds to $\theta_1$, of the corresponding inner analytic function $w_\gamma(z)$, exists and is finite. This implies that $w_\gamma(z)$ may have at $z_1$ a soft singularity, but not a hard singularity.

Note that, although it is customary to list both separately, the third condition is in fact just a particular case of the fourth condition. It is also arguable that the second condition is not really necessary, because it is a consequence of the others. We may therefore consider that the only really essential conditions are the first and the fourth ones.

The functions $g(\theta)$ are sometimes named *test functions* within the Schwartz theory of distributions [2, 3]. The additional part of the fourth condition, that the limit to the point $z_1$ of the corresponding inner analytic function $w_\gamma(z)$ must exist and be finite, consists of a weak limitation on these test functions, and does not affect the definition of the singular distribution itself. This is certainly the case for our definition here, since we define this object through a definite and unique inner analytic function.

In this section we will prove the following theorem.

**Theorem 1:** *The $\rho \to 1_{(-)}$ limit of the real part of the inner analytic function $w_\delta(z, z_1)$ converges to the generalized function $\delta(\theta - \theta_1)$.*

Before we attempt to prove this theorem, our first task is to write explicitly the real and imaginary parts of $w_\delta(z, z_1)$. In order to do this we must now rationalize it,

$$\begin{aligned} w_\delta(z, z_1) &= \frac{1}{2\pi} - \frac{1}{\pi}\frac{z(z^* - z_1^*)}{(z - z_1)(z^* - z_1^*)} \\ &= \frac{1}{2\pi} - \frac{1}{\pi}\frac{\rho^2 - \rho\cos(\Delta\theta) - \imath\rho\sin(\Delta\theta)}{\rho^2 - 2\rho\cos(\Delta\theta) + 1}, \end{aligned}$$

where $\Delta\theta = \theta - \theta_1$. We must examine the real part of this function,

$$u_\delta(\rho, \theta, \theta_1) = \frac{1}{2\pi} - \frac{1}{\pi}\frac{\rho\,[\rho - \cos(\Delta\theta)]}{(\rho^2 + 1) - 2\rho\cos(\Delta\theta)}.$$

We are now ready to prove the theorem, which we will do by simply verifying all the properties of the Dirac delta "function".

**Proof 1.1:**

If we take the limit $\rho \to 1_{(-)}$, under the assumption that $\Delta\theta \neq 0$, we get

$$\begin{aligned} \lim_{\rho \to 1_{(-)}} u_\delta(\rho, \theta, \theta_1) &= \frac{1}{2\pi} - \frac{1}{\pi} \frac{1 - \cos(\Delta\theta)}{2 - 2\cos(\Delta\theta)} \\ &= 0, \end{aligned}$$

which is the correct value for the case of the Dirac delta "function". Thus we see that the first condition is satisfied.

If, on the other hand, we calculate $u_\delta(\rho, \theta, \theta_1)$ for $\Delta\theta = 0$ and $\rho < 1$ we obtain

$$\begin{aligned} u_\delta(\rho, \theta_1, \theta_1) &= \frac{1}{2\pi} - \frac{1}{\pi} \frac{\rho(\rho - 1)}{(\rho - 1)^2} \\ &= \frac{1}{2\pi} - \frac{1}{\pi} \frac{\rho}{\rho - 1}, \end{aligned}$$

which diverges to positive infinity as $\rho \to 1_{(-)}$, as it should in order to represent the singular Dirac delta "function". This establishes that the second condition is satisfied.

We now calculate the real integral of $u_\delta(\rho, \theta, \theta_1)$ over the circle of radius $\rho < 1$, which is given by

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta\, \rho\, u_\delta(\rho, \theta, \theta_1) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \rho \left\{ 1 - \frac{2\rho\,[\rho - \cos(\Delta\theta)]}{(\rho^2 + 1) - 2\rho\cos(\Delta\theta)} \right\} \\ &= \frac{\rho}{2\pi} \int_{-\pi}^{\pi} d(\Delta\theta)\, \frac{(1 - \rho^2)}{(\rho^2 + 1) - 2\rho\cos(\Delta\theta)} \\ &= \frac{(1 - \rho^2)}{4\pi} \int_{-\pi}^{\pi} d(\Delta\theta)\, \frac{1}{[(\rho^2 + 1)/(2\rho)] - \cos(\Delta\theta)}, \end{aligned} \tag{2}$$

since $d(\Delta\theta) = d\theta$. This real integral over $\Delta\theta$ can be calculated by residues. We introduce an auxiliary complex variable $\xi = \lambda \exp(\imath\Delta\theta)$, which becomes simply $\exp(\imath\Delta\theta)$ on the unit circle $\lambda = 1$. We have $d\xi = \imath\xi d(\Delta\theta)$, and so we may write the integral on the right-hand side as

$$\begin{aligned} \int_{-\pi}^{\pi} d(\Delta\theta)\, \frac{1}{[(1 + \rho^2)/(2\rho)] - \cos(\Delta\theta)} &= \oint_C d\xi\, \frac{1}{\imath\xi}\, \frac{2}{[(1 + \rho^2)/\rho] - \xi - 1/\xi} \\ &= 2\imath \oint_C d\xi\, \frac{1}{1 - [(1 + \rho^2)/\rho]\,\xi + \xi^2}, \end{aligned}$$

where the integral is now over the unit circle $C$ in the complex $\xi$ plane. The two roots of the quadratic polynomial on $\xi$ in the denominator are given by

$$\begin{aligned} \xi_{(+)} &= 1/\rho, \\ \xi_{(-)} &= \rho. \end{aligned}$$

Since $\rho < 1$, only the simple pole corresponding to $\xi_{(-)}$ lies inside the integration contour, so we get for the integral

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta\, \frac{1}{[(1 + \rho^2)/(2\rho)] - \cos(\Delta\theta)} &= 2\imath(2\pi\imath) \lim_{\xi \to \rho} \frac{1}{\xi - 1/\rho} \\ &= 4\pi\, \frac{\rho}{(1 - \rho^2)}. \end{aligned}$$

It follows that we have for the real integral in Equation (2)

$$\begin{aligned}\int_{-\pi}^{\pi} d\theta\,\rho\,u_\delta(\rho,\theta,\theta_1) &= \frac{\left(1-\rho^2\right)}{4\pi}\,4\pi\,\frac{\rho}{\left(1-\rho^2\right)}\\ &= \rho,\end{aligned}$$

and thus we have that the integral is equal to 1 in the $\rho \to 1_{(-)}$ limit. Once we have this result, and since according to the first condition the integrand goes to zero everywhere on the unit circle except at $\Delta\theta = 0$, which is the same as $\theta = \theta_1$, the integral can be changed to one over any open interval $(a, b)$ on the unit circle containing the point $\theta_1$, without any change in its limiting value. This establishes that the third condition is satisfied.

In order to establish the validity of the fourth and last condition, we consider an essentially arbitrary integrable real function $g(\theta)$, with the additional restriction that it be continuous at the point $z_1$. As was established in [1], it corresponds to an inner analytic function

$$w_\gamma(z) = u_\gamma(\rho,\theta) + \imath v_\gamma(\rho,\theta),$$

where we also assume that $g(\theta)$ is such that $w_\gamma(z)$ may have at $z_1$ a soft singularity, but not a hard singularity, so that its limit to $z_1$ exists. We now consider the following real integral over the circle of radius $\rho < 1$,

$$\begin{aligned}\int_{-\pi}^{\pi} d\theta\,\rho\,u_\gamma(\rho,\theta)u_\delta(\rho,\theta,\theta_1) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\,\rho\,u_\gamma(\rho,\theta)\left\{1 - \frac{2\rho\left[\rho-\cos(\Delta\theta)\right]}{\left(\rho^2+1\right)-2\rho\cos(\Delta\theta)}\right\}\\ &= \frac{\rho}{2\pi}\int_{-\pi}^{\pi} d(\Delta\theta)\,u_\gamma(\rho,\theta)\,\frac{\left(1-\rho^2\right)}{\left(\rho^2+1\right)-2\rho\cos(\Delta\theta)}\\ &= \frac{\left(1-\rho^2\right)}{4\pi}\int_{-\pi}^{\pi} d(\Delta\theta)\,\frac{u_\gamma(\rho,\theta)}{\left[\left(\rho^2+1\right)/(2\rho)\right]-\cos(\Delta\theta)},\end{aligned} \tag{3}$$

since $d(\Delta\theta) = d\theta$. This real integral over $\Delta\theta$ can be calculated by residues, exactly like the one in Equation (2) which appeared before in the case of the third condition. The calculation is exactly the same except for the extra factor of $u_\gamma(\rho,\theta)$ to be taken into consideration when calculating the residue, so that we may write directly that

$$\begin{aligned}\int_{-\pi}^{\pi} d(\Delta\theta)\,\frac{u_\gamma(\rho,\theta)}{\left[\left(\rho^2+1\right)/(2\rho)\right]-\cos(\Delta\theta)} &= 2\imath(2\pi\imath)\lim_{\xi\to\rho}\frac{u_\gamma(\rho,\theta)}{\xi-1/\rho}\\ &= 4\pi\,\frac{\rho}{\left(1-\rho^2\right)}\lim_{\xi\to\rho}u_\gamma(\rho,\theta).\end{aligned}$$

Note now that since $\xi = \lambda\exp(\imath\Delta\theta)$, and since we must take the limit $\xi \to \rho$, we in fact have that in that limit

$$\lambda\,\mathrm{e}^{\imath\Delta\theta} = \rho,$$

which implies that $\lambda = \rho$ and that $\Delta\theta = 0$, and therefore that $\theta = \theta_1$. We must therefore write $u_\gamma(\rho,\theta)$ at the point given by $\rho$ and $\theta_1$, thus obtaining

$$\int_{-\pi}^{\pi} d(\Delta\theta)\,\frac{u_\gamma(\rho,\theta)}{\left[\left(\rho^2+1\right)/(2\rho)\right]-\cos(\Delta\theta)} = 4\pi\,\frac{\rho}{\left(1-\rho^2\right)}\,u_\gamma(\rho,\theta_1).$$

It follows that we have for the real integral in Equation (3)

$$\begin{aligned}\int_{-\pi}^{\pi} d\theta\,\rho\,u_\gamma(\rho,\theta)u_\delta(\rho,\theta,\theta_1) &= \frac{(1-\rho^2)}{4\pi}\,4\pi\,\frac{\rho}{(1-\rho^2)}\,u_\gamma(\rho,\theta_1)\\ &= \rho u_\gamma(\rho,\theta_1).\end{aligned}$$

Finally, we may now take the $\rho \to 1_{(-)}$ limit, since $w_\gamma(z)$ and thus $u_\gamma(\rho,\theta)$ are well defined at $z_1$ in that limit, and thus obtain

$$\begin{aligned}\lim_{\rho\to 1_{(-)}} \int_{-\pi}^{\pi} d\theta\,\rho\,u_\gamma(\rho,\theta)u_\delta(\rho,\theta,\theta_1) &= u_\gamma(1,\theta_1) \quad\Rightarrow\\ \int_{-\pi}^{\pi} d\theta\,u_\gamma(1,\theta)\left[\lim_{\rho\to 1_{(-)}} u_\delta(\rho,\theta,\theta_1)\right] &= u_\gamma(1,\theta_1) \quad\Rightarrow\\ \int_{-\pi}^{\pi} d\theta\,g(\theta)\left[\lim_{\rho\to 1_{(-)}} u_\delta(\rho,\theta,\theta_1)\right] &= g(\theta_1),\end{aligned}$$

since $u_\gamma(\rho,\theta)$ converges to $g(\theta)$, in the $\rho \to 1_{(-)}$ limit, almost everywhere on the unit circle. Just as before, once we have this result, and since according to the first condition the integrand goes to zero everywhere on the unit circle except at $\Delta\theta = 0$, which is the same as $\theta = \theta_1$, the integral can be changed to one over any open interval on the unit circle containing the point $\theta_1$, without any change in its value. This establishes that the fourth and last condition is satisfied.

Having established all the properties, we may now write symbolically that

$$\delta(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} u_\delta(\rho,\theta,\theta_1).$$

This concludes the proof of Theorem 1.

It is important to note that, when we adopt as the *definition* of the Dirac delta "function" the $\rho \to 1_{(-)}$ limit of the real part of the inner analytic function $w_\delta(z,z_1)$, the limitations imposed on the test functions $g(\theta)$ and on the corresponding inner analytic functions $w_\gamma(z)$ become irrelevant. In fact, this definitions stands by itself, and is independent of any set of test functions. Given any integrable real function $f(\theta)$ and the corresponding inner analytic function $w(z)$ with real part $u(\rho,\theta)$, we may always assemble the real integral over a circle of radius $\rho < 1$

$$\int_{-\pi}^{\pi} d\theta\,\rho\,u(\rho,\theta)u_\delta(\rho,\theta,\theta_1),$$

which is *always* well defined within the open unit disk. It then remains to be verified only whether or not the $\rho \to 1_{(-)}$ limit of this integral exists, in order to define the corresponding integral

$$\int_{-\pi}^{\pi} d\theta\,f(\theta)\delta(\theta-\theta_1).$$

This limit may exist for functions that do not satisfy the conditions imposed on the test functions. In fact, one can do this for the real part of *any* inner analytic function, regardless of whether or not it corresponds to an integrable inner analytic function, so long as the $\rho \to 1_{(-)}$ limit of $u(\rho,\theta)$ exists almost everywhere. Whenever the $\rho \to 1_{(-)}$ limit of the integral exists, it defines the action of the delta "function" on that particular real object.

It is also interesting to observe that the Dirac delta "function", although it is not simply a conventional integrable real function, is in effect an integrable real object, even if it corresponds to an inner analytic functions that has a simple pole at $z_1$, which is a non-integrable hard singularity, with degree of hardness equal to one. This apparent contradiction is explained by the *orientation* of the pole at $z = z_1$. If we consider the real part $u_\delta(\rho, \theta)$ of the inner analytic function $w_\delta(z)$, although it is not integrable along curves arriving at the singular point from most directions, there is one direction, that of the unit circle, along which one can approach the singular point so that $u_\delta(\rho, \theta)$ is identically zero during the approach, which allows us to define its integral using the $\rho \to 1_{(-)}$ limit. The same is *not* true, for example, for the imaginary part $v_\delta(\rho, \theta)$ of the same inner analytic function, which generates the Fourier-conjugate function to the delta "function", and that diverges to infinity as $1/|z - z_1|$ when one approaches the singular point along the unit circle, thus generating a non-integrable real function in the $\rho \to 1_{(-)}$ limit.

In the development presented in [1] the real functions were represented by their Fourier coefficients, and the inner analytic functions by their Taylor coefficients. We can easily do the same here, if we observe that the inner analytic function $w_\delta(z, z_1)$ in Equation (1) is the sum of a geometric series,

$$\begin{aligned} w_\delta(z, z_1) &= \frac{1}{2\pi} + \frac{1}{\pi} \frac{z/z_1}{1 - z/z_1} \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \frac{z}{z_1} \right)^k \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} [\cos(k\theta_1) - \imath \sin(k\theta_1)]\, z^k. \end{aligned} \tag{4}$$

This power series is the Taylor series of $w_\delta(z)$ around the origin, and therefore it follows that the Taylor coefficients of this inner analytic function are given by

$$\begin{aligned} c_0 &= \frac{1}{2\pi}, \\ c_k &= \frac{\cos(k\theta_1)}{\pi} - \imath \frac{\sin(k\theta_1)}{\pi}, \end{aligned}$$

where $k \in \{1, 2, 3, \ldots, \infty\}$. Since according to the construction presented in [1] we have that $c_0 = \alpha_0/2$ and that $c_k = \alpha_k - \imath\beta_k$, we have for the Fourier coefficients of the delta "function"

$$\begin{aligned} \alpha_0 &= \frac{1}{\pi}, \\ \alpha_k &= \frac{\cos(k\theta_1)}{\pi}, \\ \beta_k &= \frac{\sin(k\theta_1)}{\pi}, \end{aligned}$$

where $k \in \{1, 2, 3, \ldots, \infty\}$. Note that these are in fact the results one obtains via the integrals defining the Fourier coefficients [12],

$$\begin{aligned} \alpha_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, \cos(k\theta) \delta(\theta - \theta_1), \\ \beta_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta\, \sin(k\theta) \delta(\theta - \theta_1), \end{aligned}$$

by simply using the fundamental property of the delta "function".

Having established the representation of the Dirac delta "function" within the structure of the inner analytic functions, in sequence we will show that the Dirac delta "function" is not the only singular distribution that can be represented by an inner analytic function. As we will see, one can do the same for its first derivative, and in fact for its derivatives of any order. This is an inevitable consequence of the fact that the proper inner analytic function $w_{\delta}^{0\bullet}(z, z_1)$ associated to $w_{\delta}(z, z_1)$ is a member of an integral-differential chain.

# 3 Derivatives of the Delta "Function"

The derivatives of the Dirac delta "function" are defined in a way which is similar to that of the delta "function" itself. The first condition is the same, and the second and third conditions are not really required. The crucial difference is that the fourth condition in the definition of the Dirac delta "function" is replaced by the second condition in the list that follows. The "function" $\delta^{n\prime}(\theta - \theta_1)$ is the $n^{\rm th}$ derivative of $\delta(\theta - \theta_1)$ with respect to $\theta$ if its defining limit $\rho \to 1_{(-)}$ satisfies the two conditions that follow.

1. The defining limit of $\delta^{n\prime}(\theta - \theta_1)$ tends to zero when one takes the $\rho \to 1_{(-)}$ limit while keeping $\theta \neq \theta_1$.

2. Given any integrable real function $g(\theta)$ which is differentiable to the $n^{\rm th}$ order, in the $\rho \to 1_{(-)}$ limit the integral

$$\int_{a}^{b} d\theta\, g(\theta)\delta^{n\prime}(\theta - \theta_1) = (-1)^n g^{n\prime}(\theta_1)$$

   has the value shown, for any open interval $(a, b)$ which contains the point $\theta_1$, where $g^{n\prime}(\theta)$ is the $n^{\rm th}$ derivative of $g(\theta)$ with respect to $\theta$.

   This is the usual form of this condition, when it is formulated in strictly real terms. However, we will impose a slight additional restriction on the real functions $g(\theta)$, by assuming that the limit to the point $z_1$ on the unit circle that corresponds to $\theta_1$, of the $n^{\rm th}$ angular derivative of the corresponding inner analytic function $w_{\gamma}(z)$, exists and is finite. Since these proper inner analytic functions are all in the same integral-differential chain, this implies that the limits to $z_1$ of all the inner analytic functions $w_{\gamma}^{m\bullet}(z)$ exist, for all $0 \leq m \leq n$.

The second condition above is, in fact, the fundamental property of each derivative of the delta "function", including the "function" itself in the case $n = 0$. Just as in the case of the delta "function" itself, the additional part of the second condition, involving the inner analytic function $w_{\gamma}(z)$, consists of a weak limitation on the test functions $g(\theta)$, and does not affect the definition of the singular distributions themselves. This is certainly the case for our definitions here, since we define each one of these objects through a definite and unique inner analytic function.

In this section we will prove the following theorem.

**Theorem 2:** *For every strictly positive integer $n$ there exists an inner analytic function $w_{\delta^{n\prime}}(z, z_1)$ whose real part, in the $\rho \to 1_{(-)}$ limit, converges to $\delta^{n\prime}(\theta - \theta_1)$.*

Before we attempt to prove this theorem, let us note that the proof relies on a property of angular differentiation, which was established in [1], namely that angular differentiation is equivalent to partial differentiation with respect to $\theta$, at constant $\rho$. When we take the $\rho \to 1_{(-)}$ limit, this translates to the fact that taking the angular derivative of the inner analytic function $w(z)$ within the open unit disk corresponds to taking the derivative with respect to $\theta$, on the unit circle, of the corresponding real object.

If this derivative cannot be taken directly on the unit circle, then one can *define* it by taking the angular derivative of the corresponding inner analytic function and then considering the $\rho \to 1_{(-)}$ limit of the real part of the resulting function. Since analytic functions can be differentiated any number of times, the procedure can then be iterated in order to define all the higher-order derivatives with respect to $\theta$ on the unit circle. Equivalently, one can just consider traveling along the integral-differential chain indefinitely in the differentiation direction.

Consider therefore the integral-differential chain of proper inner analytic functions that is obtained from the proper inner analytic function associated to $w_\delta(z, z_1)$, that is, the unique integral-differential chain to which the proper inner analytic function

$$\begin{aligned} w_\delta^{0\bullet}(z, z_1) &= w_\delta(z, z_1) - \frac{1}{2\pi} \\ &= -\frac{1}{\pi} \frac{z}{z - z_1} \end{aligned} \tag{5}$$

belongs. Consider in particular the set of proper inner analytic functions which is obtained from $w_\delta^{0\bullet}(z, z_1)$ in the differentiation direction along this chain, for which we have

$$\begin{aligned} w_\delta^{n\bullet}(z, z_1) &= u_\delta^{n\prime}(\rho, \theta, \theta_1) + \imath v_\delta^{n\prime}(\rho, \theta, \theta_1) \\ &= \frac{\partial^n}{\partial\theta^n} u_\delta(\rho, \theta, \theta_1) + \imath \frac{\partial^n}{\partial\theta^n} v_\delta(\rho, \theta, \theta_1), \end{aligned}$$

for all strictly positive $n$, where we recall that

$$w_\delta(z, z_1) = u_\delta(\rho, \theta, \theta_1) + \imath v_\delta(\rho, \theta, \theta_1).$$

We will now prove that in the $\rho \to 1_{(-)}$ limit we have

$$\delta^{n\prime}(\theta - \theta_1) = \lim_{\rho \to 1_{(-)}} u_\delta^{n\prime}(\rho, \theta, \theta_1), \tag{6}$$

for $n \in \{1, 2, 3, \ldots, \infty\}$, or, equivalently, that we have for the inner analytic function $w_{\delta^{n\prime}}(z, z_1)$ associated to the derivative $\delta^{n\prime}(\theta - \theta_1)$

$$w_{\delta^{n\prime}}(z, z_1) = w_\delta^{n\bullet}(z, z_1),$$

for $n \in \{1, 2, 3, \ldots, \infty\}$. We are now ready to prove the theorem, as stated in Equation (6). Let us first prove, however, that the first conditions holds for all the derivatives of the delta "function".

**Proof 2.1:**

Since $w_\delta(z, z_1)$ has a *single* singular point at $z_1$, the same is true for all its angular derivatives. Therefore the $\rho \to 1_{(-)}$ limit of all the angular derivatives exists everywhere within the open interval of the unit circle that excludes the point $\theta_1$. Since $u_\delta(1, \theta, \theta_1)$ is identically zero within this interval, and since angular differentiation within the open unit disk corresponds to differentiation with respect to $\theta$ on the unit circle, so that we have

$$u_\delta^{n\prime}(1,\theta,\theta_1) = \frac{\partial^n}{\partial\theta^n} u_\delta(1,\theta,\theta_1),$$

for $n \in \{1, 2, 3, \ldots, \infty\}$, it follows at once that

$$\begin{aligned} u_\delta^{n\prime}(1,\theta,\theta_1) &= 0 \quad \Rightarrow \\ \lim_{\rho\to 1_{(-)}} u_\delta^{n\prime}(\rho,\theta,\theta_1) &= 0, \end{aligned}$$

for $n \in \{1, 2, 3, \ldots, \infty\}$, everywhere but at the singular point $\theta_1$, for all values of $n$. This establishes that the first condition holds.

Let us now prove that the second condition, which relates directly to the singular point, holds, leading to the result as stated in Equation (6).

**Proof 2.2:**

In order to do this, we start with the case $n = 1$, and consider the following real integral on the circle of radius $\rho < 1$, which we integrate by parts, noting that the integrated term is zero because we are integrating on a circle,

$$\int_{-\pi}^{\pi} d\theta\, u_\gamma(\rho,\theta) \left[\frac{\partial}{\partial\theta} u_\delta(\rho,\theta,\theta_1)\right] = -\int_{-\pi}^{\pi} d\theta \left[\frac{\partial}{\partial\theta} u_\gamma(\rho,\theta)\right] u_\delta(\rho,\theta,\theta_1),$$

where $w_\gamma(z) = u_\gamma(\rho,\theta) + \imath v_\gamma(\rho,\theta)$ is the inner analytic function associated to $g(\theta)$. Note that the partial derivatives involved certainly exist, since both $u_\delta(\rho,\theta,\theta_1)$ and $u_\gamma(\rho,\theta)$ are the real parts of inner analytic functions. If we now take the $\rho \to 1_{(-)}$ limit, on the right-hand side we recover the Dirac delta "function" on the unit circle, and therefore we have

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta\, g(\theta) \left[\lim_{\rho\to 1_{(-)}} \frac{\partial}{\partial\theta} u_\delta(\rho,\theta,\theta_1)\right] &= -\int_{-\pi}^{\pi} d\theta \left[\frac{d}{d\theta} g(\theta)\right] \delta(\theta-\theta_1) \\ &= (-1)\, g'(\theta_1), \end{aligned}$$

so long as $g(\theta)$ is differentiable, were we used the fundamental property of the Dirac delta "function". We thus obtain the relation for the derivative of the delta "function",

$$\int_{-\pi}^{\pi} d\theta\, g(\theta) \delta'(\theta-\theta_1) = (-1)\, g'(\theta_1),$$

where

$$\delta'(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} \frac{\partial}{\partial\theta} u_\delta(\rho,\theta,\theta_1).$$

We may therefore write that

$$\delta'(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} u_\delta'(\rho,\theta,\theta_1).$$

In this way we have obtained the result for $\delta'(\theta-\theta_1)$ by using the known result for $\delta(\theta-\theta_1)$. We may now repeat this procedure to obtain the result for $\delta^{2\prime}(\theta-\theta_1)$ from the result for $\delta'(\theta-\theta_1)$, and therefore from the result for $\delta(\theta-\theta_1)$. We simply consider the following real integral on the circle of radius $\rho < 1$, which we again integrate by parts, recalling that the integrated term is zero because we are integrating on a circle,

$$\int_{-\pi}^{\pi} d\theta\, u_\gamma(\rho,\theta) \left[\frac{\partial}{\partial\theta} u'_\delta(\rho,\theta,\theta_1)\right] = - \int_{-\pi}^{\pi} d\theta\, \left[\frac{\partial}{\partial\theta} u_\gamma(\rho,\theta)\right] u'_\delta(\rho,\theta,\theta_1).$$

If we now take the $\rho \to 1_{(-)}$ limit, on the right-hand side we recover the first derivative of the Dirac delta "function" on the unit circle, and therefore we have

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta\, g(\theta) \left[\lim_{\rho\to 1_{(-)}} \frac{\partial}{\partial\theta} u'_\delta(\rho,\theta,\theta_1)\right] &= -\int_{-\pi}^{\pi} d\theta\, \left[\frac{d}{d\theta} g(\theta)\right] \delta'(\theta-\theta_1) \\ &= (-1)^2 g^{2\prime}(\theta_1), \end{aligned}$$

so long as $g(\theta)$ is differentiable to second order, were we used the fundamental property of the first derivative of the Dirac delta "function". We thus obtain the relation for the second derivative of the delta "function",

$$\int_{-\pi}^{\pi} d\theta\, g(\theta) \delta^{2\prime}(\theta-\theta_1) = (-1)^2 g^{2\prime}(\theta_1),$$

where

$$\delta^{2\prime}(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} \frac{\partial^2}{\partial\theta^2} u_\delta(\rho,\theta,\theta_1).$$

We may therefore write that

$$\delta^{2\prime}(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} u_\delta^{2\prime}(\rho,\theta,\theta_1).$$

Clearly, this procedure can be iterated $n$ times, thus resulting in the relation

$$\delta^{n\prime}(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} u_\delta^{n\prime}(\rho,\theta,\theta_1),$$

for $n \in \{1,2,3,\ldots,\infty\}$. Note that all the derivatives with respect to $\theta$ involved in the argument exist, for arbitrarily high orders, since both $u_\delta(\rho,\theta,\theta_1)$ and $u_\gamma(\rho,\theta)$ are the real parts of inner analytic functions, and thus are infinitely differentiable on both arguments.

We may now formalize the proof using finite induction. We thus assume the results for the case $n-1$,

$$\begin{aligned} \delta^{(n-1)\prime}(\theta-\theta_1) &= \lim_{\rho\to 1_{(-)}} u_\delta^{(n-1)\prime}(\rho,\theta,\theta_1), \\ \int_a^b d\theta\, g(\theta)\delta^{(n-1)\prime}(\theta-\theta_1) &= (-1)^{n-1} g^{(n-1)\prime}(\theta_1), \end{aligned}$$

and proceed to examine the next case. We consider therefore the following real integral on the circle of radius $\rho < 1$, which we integrate by parts, recalling once more that the integrated term is zero because we are integrating on a circle,

$$\int_{-\pi}^{\pi} d\theta\, u_\gamma(\rho,\theta) \left[\frac{\partial}{\partial\theta} u_\delta^{(n-1)\prime}(\rho,\theta,\theta_1)\right] = - \int_{-\pi}^{\pi} d\theta\, \left[\frac{\partial}{\partial\theta} u_\gamma(\rho,\theta)\right] u_\delta^{(n-1)\prime}(\rho,\theta,\theta_1).$$

If we now take the $\rho \to 1_{(-)}$ limit, on the right-hand side we recover the $(n-1)^{\rm th}$ derivative of the Dirac delta "function" on the unit circle, and therefore we have

$$\int_{-\pi}^{\pi} d\theta\, g(\theta) \left[ \lim_{\rho\to 1_{(-)}} \frac{\partial}{\partial\theta} u_{\delta}^{(n-1)\prime}(\rho,\theta,\theta_1) \right] = -\int_{-\pi}^{\pi} d\theta \left[ \frac{d}{d\theta} g(\theta) \right] \delta^{(n-1)\prime}(\theta-\theta_1)$$
$$= (-1)^n\, g^{n\prime}(\theta_1),$$

so long as $g(\theta)$ is differentiable to order $n$, were we used the fundamental property of the $(n-1)^{\rm th}$ derivative of the Dirac delta "function". We thus obtain the relation for the $n^{\rm th}$ derivative of the delta "function",

$$\int_{-\pi}^{\pi} d\theta\, g(\theta) \delta^{n\prime}(\theta-\theta_1) = (-1)^n\, g^{n\prime}(\theta_1),$$

where

$$\delta^{n\prime}(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} \frac{\partial^n}{\partial\theta^n} u_{\delta}(\rho,\theta,\theta_1).$$

We may therefore write that, by finite induction,

$$\delta^{n\prime}(\theta-\theta_1) = \lim_{\rho\to 1_{(-)}} u_{\delta}^{n\prime}(\rho,\theta,\theta_1),$$

for $n \in \{1,2,3,\ldots,\infty\}$. We have therefore completed the proof of Theorem 2.

It is important to note that, just as in the case of the Dirac delta "function", when we adopt as the *definition* of the $n^{\rm th}$ derivative of the delta "function" the $\rho \to 1_{(-)}$ limit of the real part of the inner analytic function $w_{\delta}^{n\bullet}(z)$, for $n \in \{1,2,3,\ldots,\infty\}$, the limitations imposed on the test functions $g(\theta)$ and on the corresponding inner analytic functions $w_{\gamma}(z)$ become irrelevant. In fact, these definitions stand by themselves, and are independent of any set of test functions. Not only one can use them for any inner analytic functions derived from integrable real functions, but one can do this for *any* inner analytic function $w(z)$, regardless of whether or not it corresponds to an integrable real function, so long as the $\rho \to 1_{(-)}$ limit of the corresponding real part $u(\rho,\theta)$ exists almost everywhere. Just as in the case of the Dirac delta "function", whenever the $\rho \to 1_{(-)}$ limit of the real integral

$$\int_{-\pi}^{\pi} d\theta\, \rho\, u(\rho,\theta) u_{\delta}^{n\prime}(\rho,\theta,\theta_1),$$

exists, it defines the action of the $n^{\rm th}$ derivative of the delta "function" on that particular real object.

It is also interesting to observe that, just as in the case of the Dirac delta "function", it is true that its derivatives of all orders, although they are not simply integrable real functions, are in fact integrable real objects, even if they are related to inner analytic functions with non-integrable hard singularities. Just as is the case for the inner analytic function associated to the delta "function" itself, the poles of the proper inner analytic functions associated to the derivatives are always oriented in such a way that one can approach the singularities along the unit circle while keeping the real parts of the functions equal to zero, a fact that allows one to define the integrals on $\theta$ of the real parts via the $\rho \to 1_{(-)}$ limit. Just as in the case of the delta "function", the Fourier-conjugate functions of the derivatives are simply non-integrable real functions. This fact provides the first hint that there must be some relation of such non-integrable real functions with corresponding inner analytic functions.

In the development presented in [1] the real functions were represented by their Fourier coefficients, and the inner analytic functions by their Taylor coefficients. The same can be done in our case here. Starting from the power series for $w_{\delta}^{0\bullet}(z)$ given in Equation (4), we can see that the definition of the angular derivative implies that we have for the inner analytic functions associated to the derivatives of the delta "function",

$$w_{\delta}^{n\bullet}(z, z_1) = \frac{1}{\pi} \sum_{k=1}^{\infty} \imath^n k^n \left[\cos(k\theta_1) - \imath \sin(k\theta_1)\right] z^k,$$

for $n \in \{1, 2, 3, \ldots, \infty\}$, so that the corresponding Taylor coefficients are given by $c_0^{(n)} = 0$ and

$$c_k^{(n)} = \frac{\imath^n k^n}{\pi} \left[\cos(k\theta_1) - \imath \sin(k\theta_1)\right],$$

for $n \in \{1, 2, 3, \ldots, \infty\}$, and where $k \in \{1, 2, 3, \ldots, \infty\}$. The identification of the Fourier coefficients $\alpha_k$ and $\beta_k$ will now depend on the parity of $n$.

Once we have the Dirac delta "function" and all its derivatives, both as inner analytic functions and as the corresponding real objects, we may consider collections of such singular objects, with their singularities located at all the possible points of the periodic interval $[-\pi, \pi]$, as well as arbitrary linear combinations of some or all of them. There is a well-known theorem of the Schwartz theory of distributions [2, 3] which states that any distribution which is singular at a given point $\theta_1$ can be expressed as a linear combination of the Dirac delta "function" $\delta(\theta - \theta_1)$ and its derivatives of arbitrarily high orders $\delta^{n\prime}(\theta - \theta_1)$.

Since, as was observed in [1], the set of all inner analytic functions forms a vector space over the field of complex numbers, it is immediately apparent that we may assemble such linear combinations within the space of inner analytic functions. Therefore, the set of distributions formed by the delta "functions" and all their derivatives, as defined here, with their singularities located at all possible points of the unit circle, constitutes a complete basis that spans the space of all possible singular Schwartz distributions defined in a compact domain. We may conclude therefore that the whole space of Schwartz distributions in a compact domain is contained within the set of inner analytic functions.

It is interesting to note that, since we have the inner analytic function that corresponds to the delta "function" in explicit form, we are in a position to perform simple calculations in order to obtain in explicit form the inner analytic functions that correspond to the first few derivatives of the delta "function". For example, a few simple and straightforward calculations lead to the following proper inner analytic functions,

$$\begin{aligned}
w_{\delta^{1\prime}}(z, z_1) &= -\frac{1}{\pi \imath^1} \frac{z z_1}{(z - z_1)^2}, \\
w_{\delta^{2\prime}}(z, z_1) &= -\frac{1}{\pi \imath^2} \frac{z(z + z_1) z_1}{(z - z_1)^3}, \\
w_{\delta^{3\prime}}(z, z_1) &= -\frac{1}{\pi \imath^3} \frac{z\left(z^2 + 4 z z_1 + z_1^2\right) z_1}{(z - z_1)^4}.
\end{aligned}$$

These proper inner analytic functions are all very simple rational functions of the complex variable $z$, which can be written as functions of only $z/z_1$, and hence as functions of only $\rho$ and $\theta - \theta_1$. Note that we can induce from these examples that the $n^{\text{th}}$ derivative of the delta "function" is indeed represented by an inner analytic function with a pole of order $n + 1$ on the unit circle, which is thus a hard singularity with degree of hardness $n + 1$, as one would expect from the structure of the corresponding integral-differential chain.

# 4 Piecewise Polynomial Functions

It is important to note that the Dirac delta "function" and all its derivatives, with their singularities located at a given point $z_1$ on the unit circle, are all contained within a single integral-differential chain, making up, in fact, only a part of that chain, the semi-infinite chain starting from the delta "function" and propagating indefinitely in the differentiation direction along the chain. However, the chain propagates to infinity in both directions. In order to complete its analysis, we must now determine what is the character of the real objects in the remaining part of that chain, in the integration direction. In fact, they are just integrable real functions, although they do have a specific character. They consist of sections of polynomials wrapped around the unit circle, of progressively higher orders, and progressively smoother across the singular point, as functions of $\theta$, as one goes along the integral-differential chain in the integration direction.

Let us illustrate this fact with a few simple examples. Instead of performing angular integrations of the inner analytic functions, we will do this by performing integrations on the unit circle. As was established in [1], one can determine these functions by simple integration on $\theta$, so long as one remembers two things: first, to make sure that the real functions or related objects to be integrated on $\theta$ have zero average over the unit circle, and second, to choose the integration constant so that the resulting real functions also have zero average over the unit circle. For example, the integral of the zero-average delta "function"

$$\delta^{0\prime}(\theta - \theta_1) = \delta(\theta - \theta_1) - \frac{1}{2\pi},$$

which is obtained from the real part of the proper inner analytic function in Equation (5), can be integrated by means of the simple use of the fundamental property of the delta "function", thus yielding

$$\begin{aligned} \delta^{-1\prime}(\Delta\theta) &= \frac{1}{2} - \frac{\Delta\theta}{2\pi} \quad \text{for} \quad \Delta\theta > 0, \\ \delta^{-1\prime}(\Delta\theta) &= -\frac{1}{2} - \frac{\Delta\theta}{2\pi} \quad \text{for} \quad \Delta\theta < 0, \end{aligned}$$

where $\Delta\theta = \theta - \theta_1$. This is a sectionally linear function, with a single section consisting of the intervals $[-\pi, 0)$ and $(0, \pi]$, thus excluding the point $\Delta\theta = 0$ where the singularity lies, and with a unit-height step discontinuity at that point. Note that it is an odd function of $\Delta\theta$. The next case can now be calculated by straightforward integration, which yields

$$\begin{aligned} \delta^{-2\prime}(\Delta\theta) &= -\frac{\pi}{6} + \frac{\Delta\theta}{2} - \frac{\Delta\theta^2}{4\pi} \quad \text{for} \quad \Delta\theta > 0, \\ \delta^{-2\prime}(\Delta\theta) &= -\frac{\pi}{6} - \frac{\Delta\theta}{2} - \frac{\Delta\theta^2}{4\pi} \quad \text{for} \quad \Delta\theta < 0. \end{aligned}$$

This is a sectionally quadratic function, this time a continuous function, again with the same single section excluding the point $\Delta\theta = 0$, but now with a point of non-differentiability there. Note that it is an even function of $\Delta\theta$. The next case yields, once more by straightforward integration,

$$\begin{aligned} \delta^{-3\prime}(\Delta\theta) &= -\frac{\pi\Delta\theta}{6} + \frac{\Delta\theta^2}{4} - \frac{\Delta\theta^3}{12\pi} \quad \text{for} \quad \Delta\theta > 0, \\ \delta^{-3\prime}(\Delta\theta) &= -\frac{\pi\Delta\theta}{6} - \frac{\Delta\theta^2}{4} - \frac{\Delta\theta^3}{12\pi} \quad \text{for} \quad \Delta\theta < 0. \end{aligned}$$

This is a sectionally cubic continuous and differentiable function, again with the same single section excluding the point $\Delta\theta = 0$. Note that it is an odd function of $\Delta\theta$. The trend is now quite clear. All the real functions in the chain, in the integration direction starting from the delta "function", are what we may call *piecewise polynomials*, even if we have just a single piece within a single section of the unit circle, as is the case here. The $n^{\rm th}$ integral is a piecewise polynomial of order $n$, which has zero average over the unit circle, and which becomes progressively smoother across the singular point as one goes along the integral-differential chain in the integration direction.

In order to generalize this analysis, we must now consider linear superpositions of delta "functions" and derivatives of delta "functions", with their singularities situated at various points on the unit circle. A simple example of such a superposition, which we may use to illustrate what happens when we make one, is that of two delta "functions", with singularities at $\theta = 0$ and at $\theta = \pm\pi$, added together with opposite signs,

$$f(\theta) = \delta(\theta) - \delta(\theta - \pi),$$

that corresponds to the following inner analytic function, which this time is already a proper inner analytic function, with two simple poles at $z = \pm 1$,

$$\begin{aligned} w(z) &= -\frac{1}{\pi}\frac{z}{z-1} + \frac{1}{\pi}\frac{z}{z+1} \\ &= -\frac{2}{\pi}\frac{z}{z^2-1}. \end{aligned}$$

Since we have now two singular points, one at $z = 1$ and another at $z = -1$, corresponding respectively to $\theta = 0$ and $\theta = \pm\pi$, we have now two sections, one in $(-\pi, 0)$ and another in $(0, \pi)$. The inner analytic functions at the integration side of the integral-differential chain to which this function belongs are obtained by simply adding the corresponding inner analytic functions at the integration sides of the integral-differential chains of the two functions that are superposed. The same is true for the corresponding real objects within each section of the unit circle. Since the real functions corresponding to each one of the two delta "functions" that were superposed are zero-average piecewise polynomials, so are the real functions corresponding to the superposition. For example, it is not difficult to show that the first integral is the familiar square wave, with amplitude $1/2$,

$$\begin{aligned} f^{-1\prime}(\theta) &= \frac{1}{2} \quad \text{for} \quad \theta > 0, \\ f^{-1\prime}(\theta) &= -\frac{1}{2} \quad \text{for} \quad \theta < 0, \end{aligned}$$

which is a piecewise linear function with two sections, having unit-height step discontinuities with opposite signs at the two singular points $\theta = 0$ and $\theta = \pm\pi$.

We want to determine what is the character of the real functions, in the integration side of the resulting integral-differential chain, in the most general case, when we consider arbitrary linear superpositions of a finite number of delta "functions" and derivatives of delta "functions", with their singularities situated at various points on the unit circle. From the examples we see that, when we superpose several singular distributions with their singularities at various points, the complete set of all the singular points defines a new set of sections. Given one of these singular points, since at least one of the distributions being superposed is singular at that point, in general so is the superposition. Let there be $N \geq 1$ singular points $\{\theta_1, \ldots, \theta_N\}$ in the superposition. It follows that in general we end up with a set of $N$ contiguous sections, consisting of open intervals between singular points, that can be represented as the sequence

$$\left\{(\theta_1, \theta_2), \ldots, (\theta_{i-1}, \theta_i), (\theta_i, \theta_{i+1}), \ldots, (\theta_N, \theta_1)\right\},$$

where we see that the sequence goes around the unit circle, and where we adopt the convention that each section $(\theta_i, \theta_{i+1})$ is numbered after the singular point $\theta_i$ at its left end. In addition to this, since for each one of the distributions being superposed the real functions on the integration side of the integral-differential chain of the corresponding delta "function" are piecewise polynomials, and since the sum of any finite number of polynomials is also a polynomial, so are the real functions of the integral-differential chain to which the superposition belongs, if we are at a point in that integral-differential chain where all singular distributions have already been integrated out. Let us establish a general notation for these piecewise polynomial real functions, as well as a formal definition for them.

**Definition 1:** *Piecewise Polynomial Real Functions*

Given a real function $f_{(n)}(\theta)$ that is defined in a piecewise fashion by polynomials in $N \geq 1$ sections of the unit circle, with the exclusion of a finite set of $N$ singular points $\theta_i$, with $i \in \{1, \ldots, N\}$, so that the polynomial $P_i^{(n_i)}(\theta)$ at the $i^{\rm th}$ section has order $n_i$, we denote the function by

$$f_{(n)}(\theta) = \left\{ P_i^{(n_i)}(\theta), i \in \{1, \ldots, N\} \right\},$$

where $n$ is the largest order among all the $N$ orders $n_i$. We say that $f_{(n)}(\theta)$ is a *piecewise polynomial real function* of order $n$.

Note that, being made out of finite sections of polynomials, the real function $f_{(n)}(\theta)$ is always an integrable real function. In fact, it is also analytic within each section, so that the $N$ singularities described above are the *only* singularities involved. Since $f_{(n)}(\theta)$ is an integrable real function, let $w(z)$ be the inner analytic function that corresponds to this integrable real function, as constructed in [1]. The $(n+1)^{\rm th}$ angular derivative of $w(z)$ is the inner analytic function $w^{(n+1)\bullet}(z)$, which corresponds therefore to the $(n+1)^{\rm th}$ derivative of $f_{(n)}(\theta)$ with respect to $\theta$, that we denote by $f_{(n)}^{(n+1)\prime}(\theta)$.

In this section we will prove the following theorem.

**Theorem 3:** *If the real function $f_{(n)}(\theta)$ is a non-zero piecewise polynomial function of order $n$, defined in $N \geq 1$ sections of the unit circle, with the exclusion of a finite non-empty set of $N$ singular points $\theta_i$, then and only then the derivative $f_{(n)}^{(n+1)\prime}(\theta)$ is the superposition of a non-empty set of delta "functions" and derivatives of delta "functions" on the unit circle, with the singularities located at some of the points $\theta_i$, and of nothing else.*

**Proof 3.1:**

In order to prove this, first let us note that the derivative $f_{(n)}^{(n+1)\prime}(\theta)$ is identically zero within all the open intervals defining the sections. This is so because the maximum order of all the polynomials involved is $n$, and the $(n+1)^{\rm th}$ derivative of a polynomial of order equal to or less than $n$ is identically zero,

$$f_{(n)}^{(n+1)\prime}(\theta) = 0 \quad \text{for all} \quad \theta \neq \theta_i, i \in \{1, \ldots, N\}.$$

We conclude, therefore, that the real object represented by the inner analytic function $w^{(n+1)\bullet}(z)$ has support only at the $N$ isolated singular points $\theta_i$, thus implying that it can contain only singular distributions.

Second, let us prove that the derivative cannot be identically zero over the whole unit circle. In order to do this we note that one cannot have a non-zero piecewise polynomial real function of order $n$, such as the one described above, that is also continuous and differentiable to the order $n$ on the whole unit circle. This is so because this hypothesis would lead to an impossible integral-differential chain.

If this were possible, then starting from a non-zero real function $f_{(n)}(\theta)$ that corresponds to a non-zero inner analytic function $w(z)$, and after a finite number $n$ of steps along the differentiation direction of the corresponding integral-differential chain, one would arrive at a real function that is *continuous* over the whole unit circle, that is *constant within each section* and that has *zero average* over the whole unit circle. It follows that such a function would have to be identically zero, thus corresponding to the inner analytic function $w(z) \equiv 0$. But this is not possible, because this inner analytic function belongs to another chain, the one which is constant, all members being $w(z) \equiv 0$, and we have shown in [1] that two different integral-differential chains cannot intersect.

It follows that $f_{(n)}(\theta)$ can be globally differentiable at most to order $n-1$, so that the $n^{\rm th}$ derivative is a discontinuous function, and therefore the $(n+1)^{\rm th}$ derivative already gives rise to singular distributions. Therefore, every real function that is piecewise polynomial on the unit circle, of order $n$, when differentiated $n+1$ times, so that it becomes zero within the open intervals corresponding to the existing sections, will always result in the superposition of some non-empty set of singular distributions with their singularities located at the points between two consecutive sections.

We can also establish that *only* functions of this form give rise to such superpositions of singular distributions and of nothing else. The necessity of the fact that the real functions on integral-differential chains generated by superpositions of singular distributions must be piecewise polynomials comes directly from the fact that all such distributions and all such superpositions of distributions are zero almost everywhere, in fact everywhere but at their singular points. Due to this, it is necessary that these real functions, upon a *finite* number $n+1$ of differentiations, become zero everywhere strictly within the sections, that is, within the open intervals between two successive singularities. Therefore, within each open interval the condition over the sectional real function at that interval is that

$$\frac{d^{(n+1)}}{d\theta^{(n+1)}} f_i(\theta) \equiv 0,$$

and the general solution of this ordinary differential equation of order $n+1$ is a polynomial of order $n_i \leq n$, containing at most $n+1$ non-zero arbitrary constants,

$$f_i(\theta) = P_i^{(n_i)}(\theta).$$

Since only polynomials have the property of becoming identically zero after a finite number of differentiations, it is therefore an absolute necessity that these real functions be polynomials within each one of the sections. This completes the proof of Theorem 3.

Note that the inner analytic function $w^{(n+1)\bullet}(z)$ corresponding to $f_{(n)}^{(n+1)\prime}(\theta)$ represents therefore the superposition of a non-empty set of singular distributions with their singularities located at the singular points. In other words, after $n+1$ angular differentiations of $w(z)$, which correspond to $n+1$ straight differentiations with respect to $\theta$ of the polynomials $P_i^{(n_i)}(\theta)$ within the sections, one is left with an inner analytic function $w^{(n+1)\bullet}(z)$ whose real part converges to zero in the $\rho \to 1_{(-)}$ limit, at all points on the unit circle which are not one of the $N$ singular points.

It is interesting to note that, since we have the inner analytic function that corresponds to the Dirac delta "function" in explicit form, it is not difficult to calculate directly its first angular primitive. A few simple and straightforward calculations lead to

$$\begin{aligned} w_\delta^{-1\bullet}(z,z_1) &= \frac{\imath}{\pi}\int_0^z dz'\,\frac{z'}{z'-z_1} \\ &= \frac{\imath}{\pi}\ln\left(\frac{z_1-z}{z_1}\right). \end{aligned}$$

This inner analytic function has a logarithmic singularity at $z_1$, which is a borderline hard singularity. Note that, as expected, we have that $w_\delta^{-1\bullet}(0,z_1)=0$.

# 5 Products of Distributions

In the Schwartz theory of distributions one important theorem states that it is not possible to define, in a general way, the product of two distributions [13], which has the effect that the space of Schwartz distributions cannot be promoted from a vector space to an algebra. In this section we will interpret this important fact in the context of the representation of integrable real functions and singular distributions in terms of inner analytic functions. We start by noting that, although it is always possible to define the product of two inner analytic functions, which is always an inner analytic function itself, this does *not* correspond to the product of the two corresponding real functions or related objects. If we have two inner analytic functions given by

$$\begin{aligned} w_1(z) &= u_1(\rho,\theta)+\imath v_1(\rho,\theta), \\ w_2(z) &= u_2(\rho,\theta)+\imath v_2(\rho,\theta), \end{aligned} \qquad (7)$$

the product of the two inner analytic functions is given by

$$\begin{aligned} w(z) &= [u_1(\rho,\theta)u_2(\rho,\theta)-v_1(\rho,\theta)v_2(\rho,\theta)] \\ &\quad +\imath\,[u_1(\rho,\theta)v_2(\rho,\theta)+v_1(\rho,\theta)u_2(\rho,\theta)]\,, \end{aligned}$$

whose real part is *not* just the product $u_1(\rho,\theta)u_2(\rho,\theta)$. In fact, the problem of finding an inner analytic function whose real part is this quantity often has no solution. One can see this very simply by observing that both $u_1(\rho,\theta)$ and $u_2(\rho,\theta)$ are always harmonic functions, and that the product of two harmonic functions in general is *not* a harmonic function. Since the real and imaginary parts of an inner analytic function are always harmonic functions, it follows that the problem posed in this way cannot be solved in general. The only simple case in which we can see that the problem has a solution is that in which one of the two functions being multiplied is a constant function.

Let us state in a general way the problem of the definition of the product of two distributions. Suppose that we have two inner analytic functions such as those in Equation (7). The two corresponding real objects are $u_1(1,\theta)$ and $u_2(1,\theta)$, and their product, assuming that it can be defined in strictly real terms, is simply the real object $u_1(1,\theta)u_2(1,\theta)$. The problem of finding an inner analytic function that corresponds to this product is the problem of finding an harmonic function $u_\pi(\rho,\theta)$ whose limit to the unit circle results in this real object,

$$u_\pi(1,\theta)=u_1(1,\theta)u_2(1,\theta).$$

If one can find such a harmonic function, then it is always possible to find its harmonic conjugate function $v_\pi(\rho,\theta)$ and therefore to determine the inner analytic function

$$w_\pi(z) = u_\pi(\rho,\theta) + \imath v_\pi(\rho,\theta),$$

which corresponds to the product of the two real objects. According to the construction presented in [1], this can always be done so long as the product $u_1(1,\theta)u_2(1,\theta)$ is an integrable real function of $\theta$. However, if $u_1(1,\theta)$ and $u_2(1,\theta)$ are singular objects that can only be defined as limits from within the open unit disk, then the product may not be definable in strictly real terms, and it may not be possible to find an inner analytic function such that the $\rho \to 1_{(-)}$ limit of its real part results in this product, interpreted in some consistent way. This is the content of the theorem that states that this cannot be done in general.

It is not too difficult to give examples of products which are not well defined. It suffices to consider the product of any two singular distributions which have their singularities at the same point on the unit circle. If one considers integrating the resulting object and using for this purpose the fundamental property of any of the two distributions involved, one can see that the integral is not well defined in the context of the definitions given here for the singular distributions. Although one cannot rule out that some other definition can be found to include some such cases, we certainly do not have one at this time.

We thus see that we are in fact unable to promote the whole space of integrable real functions and singular distributions to an algebra. However, there are some sub-spaces within which this can be done. Under some circumstances one can solve the problem of defining within the complex-analytic structure the product of two integrable real functions. This cannot be done for the whole sub-space of integrable real functions, because there is the possibility that the product of two integrable real function will not be integrable. However, if we restrict the sub-space to those integrable real functions which are also limited, then it can be done. This is so because the product of two limited integrable real functions is also a limited real function, and therefore integrable. In this way, one can find the inner analytic function that corresponds to the product, since according to the construction which was presented in [1], this can be done for any integrable real function. The resulting inner analytic function will not, however, be related in a simple way to the inner analytic functions of the two factor functions.

One case in which the product can always be defined is that of an integrable real function with a Dirac delta "function", so long as the real function is well defined at the singular point of the delta "function". Given the nature of the delta "function", this is equivalent to multiplying it by a mere real number, the value of the integrable real function at the singular point of the delta "function",

$$g(\theta)\delta(\theta-\theta_1) = g(\theta_1)\delta(\theta-\theta_1).$$

The corresponding inner analytic function is therefore given simply by $g(\theta_1)w_\delta(z,z_1)$. Similar statements are true, of course, for all the derivatives of the delta "function". Therefore, in all such cases there is no difficulty in determining the inner analytic function that corresponds to the product.

Note that this difficulty relates only to the definition of the product of two real objects on the unit circle. As was observed before, for all the singular distributions their definition by means of inner analytic functions always provides the means to determine whether or not they can be applied to a given real object, so long as it is represented by an inner analytic function, and what results from that operation, if it is possible at all.

## 6 Conclusions and Outlook

We have extended the close and deep relationship established in a previous paper [1], between integrable real functions and complex analytic functions in the unit disk centered at the origin of the complex plane, to include singular distributions. This close relationship between, on the one hand, real functions and related objects, and on the other hand, complex analytic functions, allows one to use the powerful and extremely well-known machinery of complex analysis to deal with the real functions and related objects in a very robust way, even if these objects are very far from being analytic. The concept of integral-differential chains of proper inner analytic functions, which we introduced in that previous paper, played a central role in the analysis presented.

One does not usually associate non-differentiable, discontinuous and unbounded real functions with single analytic functions. Therefore, it may come as a bit of a surprise that, as was established in [1], *all* integrable real functions are given by the real parts of certain inner analytic functions on the open unit disk when one approaches the unit circle. This surprise is now compounded by the fact that inner analytic functions can represent singular distributions as well and, in fact, can represent what may be understood as a complete set of such singular objects.

There are many more inner analytic functions within the open unit disk than those that were examined here and in [1], in relation to integrable real functions and singular distributions. Therefore, it may be possible to further generalize the relationship between real objects on the unit circle and inner analytic functions. For example, we have observed in this paper that there are inner analytic functions whose real parts converge to non-integrable real functions on the unit circle. Simple examples are the inner analytic functions given by

$$\bar{w}_{\delta^{n\prime}}(z, z_1) = -\imath w_{\delta^{n\prime}}(z, z_1),$$

for $n \in \{0, 1, 2, 3, \ldots, \infty\}$, that correspond to the non-integrable real functions which are the Fourier-conjugate functions of the Dirac delta "function" and its derivatives. This suggests that we consider the question of how far this can be generalized, that is, of what is the largest set of non-integrable real functions that can be represented by inner analytic functions. This issue will be discussed in the fourth paper of this series.

The singular distributions are integrable real objects associated to non-integrable singularities of the corresponding inner analytic functions, a fact which is made possible by the orientation of the singularities with respect to the direction along the unit circle. This suggests that the most general definition of such singular distributions may be formulated in terms of the type and orientation of the singularities present on the unit circle. In this case one would expect that singular distributions would be associated to inner analytic functions with hard singularities that are oriented in a particular way, so that the integrals of their real parts can be defined via limits from the open unit disk to the unit circle.

We believe that the results presented here establish a new perspective for the representation and manipulation of singular distributions. It might also constitute a simpler and more straightforward way to formulate and develop the whole theory of Schwartz distributions within a compact domain.

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