Non-Linear Scalar Models

A Continued Exploration of Quantum Field Theory

Note: this book is still incomplete; the first two chapters and two thirds of the third chapter can be found in this version; the first two chapters are complete and reasonably self-contained; the remaining material depends on stochastic simulations that are currently being worked on.

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Preface

This is the second in a series of a few short books about the foundations of quantum field theory.

We would like to acknowledge here the contribution to parts of the material contained within this book by former students and collaborators who were active in this subject in the past: Dr. Timothy Edward Gallivan and Dr. André Cavalcanti Rocha Martins. When appropriate, due credit will be given in footnotes along the text. We would also like to thank those that helped by reading and correcting the manuscript at various stages of its production.

Contents

Chapter 1

The Polynomial Model

In this chapter we will examine the polynomial models of scalar fields. These are generalizations of the Gaussian model in which one tries to introduce interactions into the model by the addition of a new term to the action, containing a power of the field which is greater than the power two that appears in the Gaussian model.

1.1 Definition of the Model

In the previous volume of this series [2] we studied in detail the theory of the free scalar field. That model was sufficiently simple to allow us to calculate analytically all the predictions of the theory. As we saw, both in the case of the classical theory and in the case of the quantum theory this simplicity follows from the linearity of the model. We also saw that this same linearity is responsible for the fact that the model does not contain the concept of interactions between particles, and hence that the only physics that it does contain is the propagation of free particles. This was shown by the factorization of all the correlation functions in terms of the propagator, and also by the fact that the energies of the particles are simply additive, that is, the energy of a state containing two particles is the sum of the energies of the two corresponding single-particle states, implying the absence of any kind of interaction energy.

We will make here a first trial at including interactions in the theory, for which it will be necessary to break the linearity of the model, including in the action terms with more that two powers of the field. We will therefore examine the model defined by

$$S[\varphi] = \frac{1}{2}\sum_{\ell}(\Delta_{\ell}\varphi)^2 + \frac{\alpha}{2}\sum_{s}\varphi^2(s) + \frac{\lambda}{4}\sum_{s}\varphi^4(s), \tag{1.1.1}$$

which we denominate the $\lambda\varphi^4$ *polynomial model*. We therefore choose to break the linearity of the model by the introduction of a new ultra-local term into the action, leaving untouched the term containing the derivatives. This is the simplest example of a model that, in the classical theory, contains interacting fields. Our task here is in the methods of stochastic simulation constitute a rather extended topic with a very technical character, and will be developed in detail in a separate volume of this small series of books.

The character of the classical theory determined by our action is clear, and it is not necessary to examine it in detail. The definition of the classical theory is the same as before, the classical solution is the configuration φ that minimizes the action. The fact that it exists is guaranteed by the conditions we impose on the parameters of the model: $\lambda > 0$ with any α or $\lambda = 0$ with $\alpha \geq 0$. We may derive, in a way analogous to the one used before for the free theory, the corresponding classical equation of motion, which will be, of course, a non-linear differential equation for φ (problem 1.1.1). In order to begin the examination of the behavior of the quantum theory, we recall that it is defined by the probability distribution over all possible configurations of φ, given by

$$\frac{[\mathbf{d}\varphi]\, e^{-\sum_\ell \frac{1}{2}(\Delta_\ell\varphi)^2-\sum_s\left(\frac{\alpha}{2}\varphi^2+\frac{\lambda}{4}\varphi^4\right)}}{\int[\mathbf{d}\varphi]\, e^{-\sum_\ell \frac{1}{2}(\Delta_\ell\varphi)^2-\sum_s\left(\frac{\alpha}{2}\varphi^2+\frac{\lambda}{4}\varphi^4\right)}},$$

where we grouped separately the term containing the derivatives and the ultra-local part, containing the polynomial terms, both the quadratic one and the quartic one, which we call the interaction term. If we recall that the measure $[\mathbf{d}\varphi]$ is a product of differentials over all the sites s, representing the fact that in this measure the stochastic variables φ have uniform probability distributions, we see that we can include the ultra-local terms in the measure, writing the distribution as

$$\frac{\left[\mathbf{d}\varphi\, e^{-\left(\frac{\alpha}{2}\varphi^2+\frac{\lambda}{4}\varphi^4\right)}\right]\, e^{-\sum_\ell \frac{1}{2}(\Delta_\ell\varphi)^2}}{\int\left[\mathbf{d}\varphi\, e^{-\left(\frac{\alpha}{2}\varphi^2+\frac{\lambda}{4}\varphi^4\right)}\right]\, e^{-\sum_\ell \frac{1}{2}(\Delta_\ell\varphi)^2}}.$$

In this new measure the variables φ no longer have uniform probability distributions, but have instead the probability distribution given by the exponential of the potential. A typical example of such a distribution can be seen in figure 1.1.1. We see in this way that a possible way to understand our model is to think of it as constituted of the dynamics implemented by the derivative term, but applied indirectly to new random variables χ located at the sites, with uniform probability distributions, which are given in terms of the variables φ by the differential relation

$$\mathrm{d}\chi = \mathrm{d}\varphi\, e^{-(\frac{\alpha}{2}\varphi^2+\frac{\lambda}{4}\varphi^4)}.$$

This means that we may write the stochastic variable χ, which has an uniform probability distribution within a closed interval, in terms of the stochastic variable φ, which has a non-uniform probability distribution over the whole real line, as

$$\chi(\varphi) = \int_0^\varphi \mathrm{d}\varphi'\, e^{-(\frac{\alpha}{2}\varphi'^2+\frac{\lambda}{4}\varphi'^4)}. \tag{1.1.2}$$

of the model at an intuitive level, it is very useful to invert the decomposition described above, representing the effect of the derivative term by the Gaussian local distribution that it implies, which was studied in detail in the first volume of this series of books, and considering directly the effects that the potential term may have on it. This allows us to use in the analysis of the model our classical intuition concerning the behavior of an object within the potential, by extending the object from a simple point body to a fluctuating statistical distribution. In this way we are able to build an almost-classical intuition in order to understand the behavior of the model.

We can imagine that we draw a copy of the potential at each site and that we put inside it the value of the field at that site, while the terms $(\Delta_\ell \varphi)^2$ interconnect each pair of neighboring sites. The stochastic dynamics implemented by these derivative terms will cause the values of the fields to fluctuate at each site, so that we will actually have a distribution of values within each potential. In the theory of the free field these local distributions are simple Gaussians, whose width σ_0 is a number of the order of 1 for any lattice size, a number that determines the value of the physical mass (also called the renormalized mass), which in the case of the free theory is simply $m_R = m_0$. In our current model the complete local distributions also have a width $\sigma \approx \sigma_0$ of the order of 1, but their format may no longer be exactly Gaussian. The renormalized mass is defined in this model, just as in the free theory, as the inverse of the correlation length of the model, measured through the two-point correlation function, and presumably it is also related to σ, as is the case for the free theory. What we propose to do in this section is to understand the complete dynamics of the model by considering directly the influence of the potential over the Gaussian local distribution implemented by the derivative term.

Let us recall that the relation between the two-point function and the renormalized mass was studied in detail, in the case of the free field, in a section of the first volume [4]. The exponential decay of the two-point function for large distances r, given by $\exp(-m_0 r)$ as was studied in that section, is a general property of all massive theories, either free or otherwise. The behavior of the two-point function of the interacting theories for large distances is in general of this same type, except for the exchange of the parameter m_0 for another parameter m_R which usually differs from m_0. In the numerical approach we usually measure this new parameter by means of a curve-fitting process applied to the numerical propagator of the theory in momentum space rather than position space, that is, to the Fourier transform of the two-point function, which is technically easier to do, and also more efficient. We do this in the expectation, to be confirmed a posteriori, that the form of this function in the interacting models is not very different from its format in the free theory, and it usually works very well.

In the model that we are introducing here, given values of N, α and λ, we will have not only a resulting value for the parameter α_R related to the renormalized mass, but also some resulting value for the *renormalized coupling constant* λ_R, which is the physical coupling constant whose nature and precise definition we will examine in more detail later on. Unlike what happened in the free theory, in general α_R will

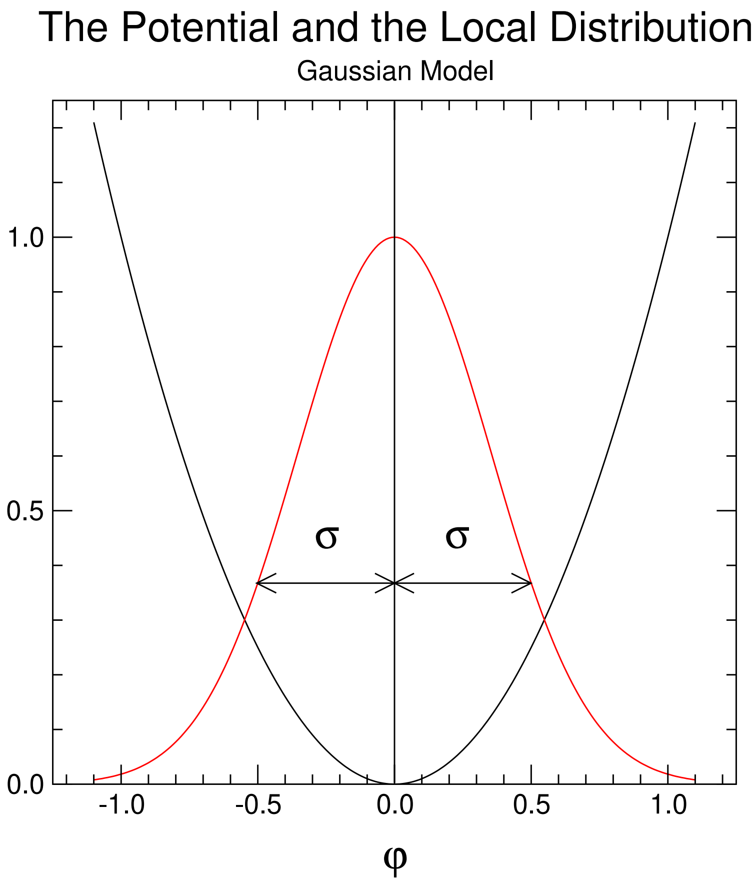

Figure 1.1.2: The potential and the local distribution due to the derivative term, in the case of the free theory.

squeezing force gives rise to an infinite m_R. Let us imagine now that we insert this local distribution inside the potential defined by the ultra-local terms of the action at each site, as shown in figure 1.1.2 for the case of the free theory, with a Gaussian distribution and a quadratic potential. If we were examining the classical theory, we would put inside the potential a point body representing the value of the classical solution for the field, and it would come to rest at the minimum of the potential. The examination of the behavior of the quantum theory corresponds to the introduction into the potential well of an extended object which can be represented heuristically by our semi-rigid body, which becomes rigid in the continuum limit in the sense explained above.

The width of the local distribution in the absence of the potential is determined only by the derivative term and corresponds to the value $\alpha_R = 0$, that is, to a zero renormalized mass. When we put the distribution under the action of the potential on a finite lattice what happens is that it tends to concentrate the values of the field around the minimum, and hence squeezes the distribution, decreasing its width, because it is statistically unfavorable for the field to exist in the positions where the

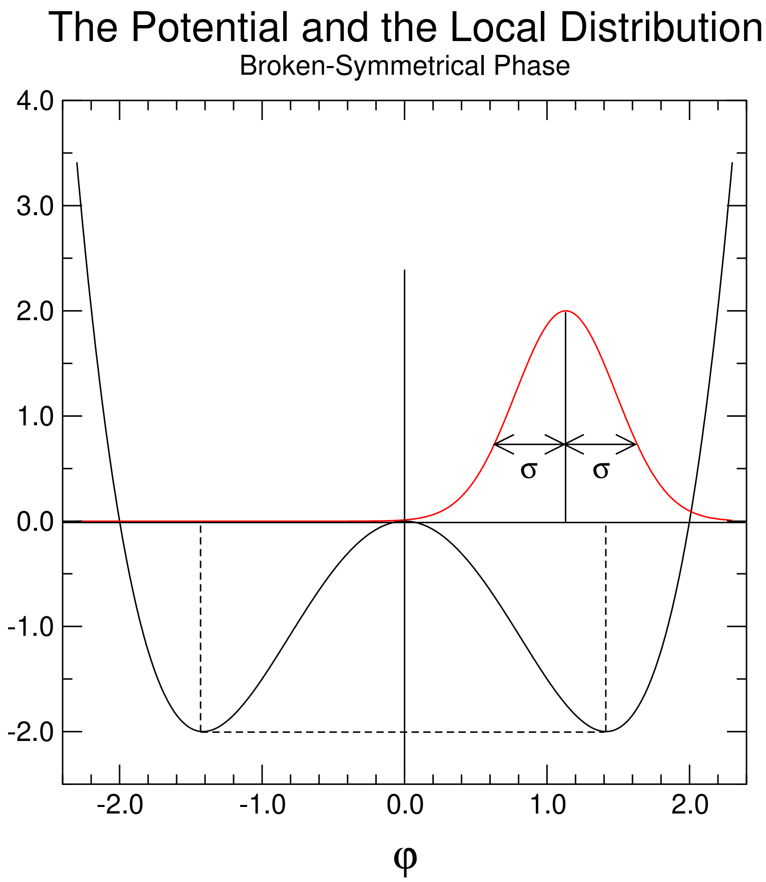

Figure 1.1.4: The potential and the local distribution due to the derivative term, in the case of the non-linear theory, in the broken-symmetrical phase.

much effort, a prediction with very serious consequences regarding the behavior of the model: if we limit ourselves to the case in which $\alpha \geq 0$ and $\lambda \geq 0$ for all N, it will be necessary to make both $\alpha \rightarrow 0$ and $\lambda \rightarrow 0$ in the limit, *which takes the model back to the critical point of the Gaussian model and therefore eliminates any possibility that λ_R be different from zero in limits of this type.*

Except for the case $d = 3$, this implies that there are in fact no interactions between particles in the quantum theory of this model, in any limits that stay within the quadrant given by $\alpha \geq 0$ and $\lambda \geq 0$. We say that in this case the model has only the *trivial* limit, leading to the theory of the free field, or that the theory is *trivial* in this sector of the space of parameters of the model. The case $d = 3$ is a little different because, since in this case the physical coupling constant has dimensions of mass, it is possible that there are interactions even if the model approaches the Gaussian point, e phenomenon that we will discuss later on.

We conclude that, if we are to have any chance of finding an interesting limit in this model, it will be necessary for at least one of the two parameters to be negative. Since we cannot make λ negative due to the stability constraints, it follows that *the*

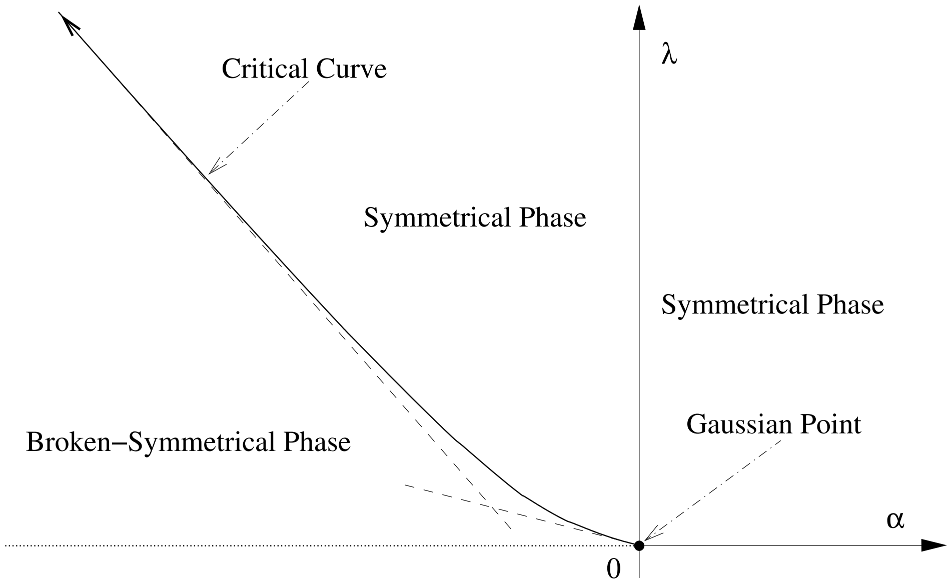

Figure 1.1.5: Critical diagram of the non-linear $\lambda\varphi^4$ model.

in this model, giving origin to two *phases* in the space of parameters of the model, in each one of which the behavior of the model is of a certain type, different from the other one. The expectation value $v_R = \langle\varphi\rangle$ is the *order parameter* of the transition, being equal to zero in one of the phases, the symmetrical one, and different from zero in the other phase, the broken-symmetrical one. Since the parameter space is part of a two-dimensional plane, we expect that the two phases be separated by a one-dimensional curve. In fact, we can easily estimate the locus of this phase-transition curve. The argument that leads us to verify that the two phases exist depends crucially on the width of the potential, which is proportional to $\sqrt{-\alpha/\lambda}$. This quantity does not change so long as α and λ are proportional to each other, $\alpha = -\beta\lambda$, so that the location of the points where the transition occurs, separating the two phases, should depend only on the proportionality constant β.

We can estimate this quantity assuming that at the transition the distance between the two wells is of the order of the width σ_0 of the local distribution, whose value is determined predominantly by the derivative term of the action. It is clear that, if the distance between the wells is significantly smaller than σ_0, the local distribution will tend to remain centered around $\varphi = 0$, with its width somewhat reduced, while if the distance is significantly larger than σ_0, the local distribution will tend to shift sideways and fall into one of the wells. In order to better understand this argument it is useful to think about the extreme cases, the one in which the total width of the potential is much smaller than σ_0 and the local distribution is highly squeezed within it, and the one in which the total width of the potential is much larger than σ_0 and the local distribution is completely free to move within the potential and therefore to fall into one of the two wells. This estimate gives us the relation $\sigma_0 \approx \sqrt{-\alpha_c/\lambda_c}$ for the critical values of the parameters or, more precisely, $C_0\sigma_0 = \sqrt{-\alpha_c/\lambda_c}$, where C_0 is some positive constant of the order of 1 that can position of the critical lines will involve some rather intensive and possibly difficult computer work.

Observe that there are many other flows that approach the Gaussian point, besides those defined directly by the free-field model. For example, we have a class of flows that go along the vertical half-axis ($\lambda > 0, \alpha = 0$) in the direction of $\lambda = 0$. This class of flows can produce trivial limits in which the resulting renormalized mass is determined by the dimensionless coupling parameter λ, instead of by α. This phenomenon, which we can see here in a very simple way, is also known by the name of "dimensional transmutation". It is also possible to approach the Gaussian point from the broken-symmetrical phase, from under the critical curve. In this way we may define trivial limits in which the field, although free, has non-zero expectation values $V_R = \langle\phi\rangle$. Except in the case $d = 3$ any limits that are candidates to not being trivial must approach some other point of the critical curve, and not the Gaussian point. If all possible flows of the model turn out to be trivial then we say that the model is trivial. Then, except for the introduction of the concept of spontaneous symmetry breaking, such models are just another way to produce the theory of the free scalar field in the continuum limit. They may still be useful test models on finite lattices, though.

Observe that what we have obtained here is a type of critical behavior just like the one described in the section in [5], where the quantity $v_R = \langle\varphi\rangle$ plays the role played by the magnetization in the case of statistical mechanics, as shown in the first figure of that section. However, we can do a little better here, and continue our heuristic argument in order to get an estimate for v_R as a function of α and λ. First of all let us point out that we should be able to get only the absolute value of v_R and not its sign, since the symmetry can break to either side. Let us therefore estimate v_R^2 and not v_R. It must be zero at the critical line, that is, when the squared width of the potential, $-\alpha/\lambda$, is equal to the squared width of the local distribution, $C_0^2\sigma_0^2$. It must also be equal to zero whenever the potential is less wide than that, that is,

$$v_R^2 = 0 \quad \text{for} \quad -\frac{\alpha}{\lambda} \leq C_0^2\sigma_0^2,$$

characterizing the symmetrical phase. To this we may add that, if the potential is wider than the width of the local distribution, then the distribution should be able to shift to one side by something like the difference between the two, giving as the corresponding estimate for v_R^2 the value

$$v_R^2 = -\frac{\alpha}{\lambda} - C_0^2\sigma_0^2 \quad \text{for} \quad -\frac{\alpha}{\lambda} \geq C_0^2\sigma_0^2,$$

characterizing the broken-symmetrical phase. Note that this formula gives the correct value $v_R^2 = 0$ at the critical line. It also gives the correct value in the case of an extremely wide potential, in which case $C_0^2\sigma_0^2$, which is always of the order of 1, can be neglected by comparison with $-\alpha/\lambda$, and we should have the average v_R of the local distribution sitting at the minimum of the potential, that is, $v_R^2 = -\alpha/\lambda$. In short, we have for v_R^2 the result point. Therefore the potential never becomes infinitely flat around its local minima in the broken-symmetrical phase, except in limits that approach the Gaussian point. There are, therefore, no other possible continuum limits with finite masses in this phase, except those that tend to the Gaussian point or to some other point of the critical line.

As a last refining touch of our argument, we may point out its relation to the question of the triviality of the model. Let us look back at figure 1.1.3 and imagine that it represents the critical situation, in which the potential is just wide enough not to squeeze the local distribution. This means that the local distribution is almost free to move, but there is no space for it to actually do so. In this situation the squeezing action of the outer walls of the potential and the spreading action of the central bump are exactly balanced, so that the net action over the width of the local distribution vanishes. However, even without changing the width, the potential can tend to change the *shape* of the distribution. In fact, if we consider in which parts of the local distribution the central bump and the outer walls act, we realize that the bump tends to flatten and spread the top of the Gaussian, while the two outer walls tend to increase the slopes on the two sides of the distribution. This change of shape is exactly what one would expect if the local distribution tended to be more like the function $\exp(-\lambda_R\varphi^4/4)$ than like a Gaussian.

In fact, one would expect λ_R to manifest itself by affecting the form of the complete local distribution. Since α_R is related to the second moment of the distribution, its width, it is reasonable to expect that λ_R is correspondingly related to the fourth moment of the distribution, and hence to the shape of the curve that describes it. However, our experience with α_R shows that, due to the derivative term of the action, the local distribution has a very rigid character in the continuum limit, as a consequence of the requirements of propagation, which requires α_R to go to zero in the limit. There is therefore legitimate doubt that we can have this distribution significantly changed in shape in the continuum limit, to allow for a non-zero value of λ_R, without disturbing the dynamics of propagation and thus ending up with infinitely massive particles and no propagation. Note that if it is true that we must have $\lambda_R \to 0$ in the limit, then it is imperative that λ_R be zero exactly over the critical line, where we already know that α_R is zero, otherwise there would be no possible continuum limits except those going to the Gaussian point.

Of course we cannot resolve this difficult matter with only heuristic arguments. In fact, we will see that perturbation theory is also not enough to handle this issue, and we will have to use computer simulations in order to explore it. However, we can say that, if indeed it turns out that we must have $\lambda_R \to 0$ in the limit, then only theories that can be non-trivial with λ_R going to zero still have a chance of being truly interacting quantum theories. This involves the scaling relations between λ_R and its dimensionfull version Λ_R. As is discussed in problem 1.1.8, in the classical case, this leaves, of all polynomial models φ^{2p}, $p = 2, 3, 4, \ldots$, in all dimensions $d = 3, 4, 5, \ldots$, a single possibility: the $\lambda\varphi^4$ model in $d = 3$.

We end this section with a short discussion of the continuum limit of the classical theory, which requires rewriting the action of the model in terms of dimensionfull the dimensionfull field ϕ, that have direct physical relevance, being directly related to the quantities which are observable in the quantum theory. Although we have only shown this fact in the case of the free theory, we will assume that it is true in general, a working hypothesis that we will only be able to confirm a posteriori by the accumulation of calculational experience, numerical or otherwise. We will see that this relation of scale between λ_R and Λ_R will be very useful to enable us to understand heuristically the behavior of the quantum models. Before anything else is done, however, it will be necessary to define in a more precise way the constant λ_R of the quantum theory, which we will do in the third chapter of this book.

Problems

1.1.1. Derive the classical equation of motion for the $\lambda\varphi^4$ model, showing that it is a non-linear equation. Write the equation both on finite lattices, in terms of the dimensionless field, and in the continuum limit, using the dimensionfull field.

1.1.2. Calculate the position, the depth and the bottom curvature of the potential wells in the models $\lambda\varphi^{2p}$, $p = 3, 4, \ldots$, which are defined by the action

$$S[\varphi] = \frac{1}{2}\sum_{\ell}(\Delta_\ell\varphi)^2 + \frac{\alpha}{2}\sum_{s}\varphi^2(s) + \frac{\lambda}{2p}\sum_{s}\varphi^{2p}(s),$$

verifying that the spontaneous symmetry breaking situation is qualitatively similar to that of the $\lambda\varphi^4$ model. Sketch qualitatively the phase diagrams of these models and estimate whether the critical curves in these cases are more or less steep than the critical curve of the quartic model.

1.1.3. Show that the stochastic variable χ introduced in the text for the $\lambda\varphi^4$ model is bound within the interval $[0, \chi_{\rm max}]$, where $\chi_{\rm max}$ is a finite quantity which depends on α and λ.

1.1.4. Define a generalization of the stochastic variable χ, which was introduced in the text for the case of the $\lambda\varphi^4$ model, for the case of the $\lambda\varphi^{2p}$, $p = 3, 4, \ldots$ models.

1.1.5. Estimate and sketch the positions, on the phase diagram of the $\lambda\varphi^4$ model, of the curves defined by $\alpha_R(\alpha, \lambda) = C$, where C is a constant. Do it on both sides of the critical curve. Remember that $\alpha_R = 0$ over the critical curve, that $\alpha_R = \alpha$ for the free theory, and that $\alpha_R \approx -2\alpha$ for negative α and $\lambda \approx 0$.

1.1.6. Write the action of the $\lambda\varphi^4$ model in terms of the Fourier transforms $\widetilde{\varphi}$ of the fields and show that, due to the presence of the quartic term, the Fourier modes in momentum space do not decouple from each other when $\lambda \neq 0$.

1.1.7. Show that, in the case $d = 1$, the model $\lambda\varphi^4$ is identical to the quantum mechanics of an anharmonic oscillator.

We may now write this in terms of the measure of the free theory defined by S_0, dividing both numerator and denominator by $\int[\mathbf{d}\varphi]e^{-S_0}$ and thus obtaining

$$\langle\mathcal{O}\rangle = \frac{\left\langle\mathcal{O}\; e^{-S_I}\right\rangle_0}{\left\langle e^{-S_I}\right\rangle_0},$$

where the subscript 0 denotes expectation values of the theory defined by S_0,

$$\langle\mathcal{O}\rangle_0 = \frac{\displaystyle\int[\mathbf{d}\varphi]\mathcal{O}[\varphi]e^{-S_0}}{\displaystyle\int[\mathbf{d}\varphi]e^{-S_0}}.$$

The term S_I of the action is the one that contains the parameter λ, that we presume to be small. However, in general S_I may contain also other parameters, so that in order to enable us to do the development of the perturbation theory in a more organized and explicit fashion it is convenient to use, instead of λ, a new expansion parameter ε that we introduce as follows,

$$f(\varepsilon) = \frac{\left\langle\mathcal{O}\; e^{-\varepsilon S_I}\right\rangle_0}{\left\langle e^{-\varepsilon S_I}\right\rangle_0}. \tag{1.2.2}$$

We have therefore that $f(0) = \langle\mathcal{O}\rangle_0$ and $f(1) = \langle\mathcal{O}\rangle$. Perturbation theory consists of making a series expansion, which we denominate the *perturbative expansion*, of $f(\varepsilon)$ around $\varepsilon = 0$, up to a certain order, followed by the use of the resulting expressions at the point $\varepsilon = 1$. Of course this can only be a good approximation to the complete theory if S_I is a small quantity. Classically we can make S_I small by adjusting the values of λ and any other parameters that it may contain but, as we shall see in what follows, *this is not possible in the quantum theory.* This is the basic fact that is at the root of all difficulties with the perturbative approach to quantum field theory.

In order to understand the origin of the difficulties it is necessary to recall some important properties of the theory of the free scalar field, since we are writing our quantities here in terms of the expectation values of that theory. As we saw in the section in [7], in the case of the dimensions $d \geq 3$ which are the ones of interest for quantum field theory, the quantity $\sigma^2 = \langle\varphi^2\rangle$, which we denote here by σ_0^2 to record the fact that it is a quantity relating to the free theory, is a finite and non-zero quantity both on finite lattices and in the continuum limit. In addition to this, we showed in the section in [8] that the quantity $\langle[\Delta_\mu\varphi]^2\rangle$ is also finite and non-zero both on finite lattices and in the limit, in which case it has the value $1/d$. Still in the section in [8] these facts were used to show that both the expectation value of the kinetic part S_K of the action,

$$S_K = \frac{1}{2}\sum_{\ell}(\Delta_\ell\varphi)^2,$$

and the expectation value of the part S_M of the action containing the mass term, trivial in that case, since the interaction term would then become vanishingly small in the continuum limit, when compared to the remaining part of the action.

The conclusion to which we are forced is that $\langle S_V\rangle_0$ in fact diverges in the continuum limit as N^d. It is important to observe once more that this divergence is not due to an integration over an infinite volume, because we can do the complete development of the theory within a finite box without any change in this result. This divergence is a property of the continuum limit, an *ultraviolet* characteristic of the theory, in which the influence of the high-frequency and short-wavelength modes of momentum space predominate. It is *not* a property of the infinite-volume limit, that is, of the *infrared* regime of the theory, in which the low-frequency and long-wavelength modes predominate. It becomes clear, therefore, that it is not possible to keep S_V small by mere changes in the parameters α and λ, except if we make them converge rapidly to zero in the continuum limit, which takes us back to the Gaussian point, where all the results are already known, constituting the theory of the free scalar field.

At this point it does not seem that this perturbative technique can end up having any practical use but, in any case, let us proceed with our analysis of the situation. If we consider for a moment the denominator of equation (1.2.1) it is clear that we will have, in the continuum limit,

$$\left\langle e^{-S_I}\right\rangle_0 \to 0,$$

while the perturbative expansion of this quantity, obtained by the series expansion of the exponential function, will contain divergent terms if we keep ε finite and non-zero when we take the limit,

$$\left\langle e^{-\varepsilon S_I}\right\rangle_0 \approx 1-\varepsilon\langle S_I\rangle_0+\ldots, \quad \text{where} \quad \langle S_I\rangle_0 \to \infty.$$

We see here that a simple and naive expansion within such a singular structure can make a vanishing quantity appear as a collection of infinities in the terms of the expansion. We can now see that the limit of equation (1.2.1) for $N\to\infty$ is a limit of the form 0/0. However, it certainly exists, so long as the theory is well defined, which we expect to be true so long as we keep the parameters of the theory within the stable region of the critical diagram. The denominator can be understood as the ratio of the measures of the interacting model and of the free theory,

$$\left\langle e^{-S_I}\right\rangle_0 = \frac{\displaystyle\int [\mathbf{d}\varphi] e^{-(S_0+S_I)}}{\displaystyle\int [\mathbf{d}\varphi] e^{-S_0}},$$

so that the conclusion we arrive at is that these two measures are related in a singular way in the continuum limit. On any finite lattice the expectation value $\langle S_I\rangle_0$ is finite and we can improve the approximation by decreasing somewhat the parameters α and λ. However, in the continuum limit the only form to avoid the divergence is to

In fact, one can treat the problem at hand by making a change in the normalization of both the numerator and the denominator of equation (1.2.1), eventually obtaining the same results that we will obtain here in a more direct way (problem 1.2.2).

We will examine here the first-order and second-order terms in ε for the expansion of $f(\varepsilon)$, for which we obtain

$$f(\varepsilon) = f(0) + \varepsilon f'(0) + \frac{1}{2}\varepsilon^2 f''(0) + \ldots,$$

where the first three terms contain (problem 1.2.3)

$$\begin{aligned} f(0) &= \langle\mathcal{O}\rangle_0, \\ f'(0) &= -\left[\langle\mathcal{O}S_I\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_I\rangle_0\right], \\ f''(0) &= \left[\langle\mathcal{O}S_I^2\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_I^2\rangle_0\right] - 2\langle S_I\rangle_0\left[\langle\mathcal{O}S_I\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_I\rangle_0\right]. \end{aligned}$$

Making $\varepsilon = 1$ we obtain

$$\begin{aligned} \langle\mathcal{O}\rangle &\approx \langle\mathcal{O}\rangle_0 - \left[\langle\mathcal{O}S_I\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_I\rangle_0\right] \\ &\quad + \frac{1}{2}\left\{\left[\langle\mathcal{O}S_I^2\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_I^2\rangle_0\right] - 2\langle S_I\rangle_0\left[\langle\mathcal{O}S_I\rangle_0 - \langle\mathcal{O}\rangle_0\langle S_I\rangle_0\right]\right\}. \end{aligned} \quad (1.2.3)$$

This is the approximation for $\langle\mathcal{O}\rangle$ up to the order ε^2, that is, effectively up to the order λ^2. We will use it later on to calculate perturbative approximations for some of the observables of the model.

Observe that it is not to be expected that this expansion may produce a convergent series for the observables of the model. An alternative way to see this is to observe that there cannot be a non-vanishing convergence radius for the series of $f(\varepsilon)$ around $\varepsilon = 0$ in the complex ε plane, because a non-vanishing convergence disk around zero would include negative values of ε, which correspond to points in the unstable region of the parameter plane of the model, where we know that it does not exist. At most what we can hope to obtain are reasonable approximations up to a certain order, which hopefully will be good enough to allow us to form a correct qualitative idea about the behavior of the model. Note that the model would be clearly more useful if it did not cease to exist when we exchange the sign of the coupling constant. One is led to recall that this is the expected situation in electrodynamics, in which we can have charges of either sign.

In order to complete the development of our perturbative ideas, we must now return to the issue of the separation of the action S in parts S_0 and S_I. This separation will depend on whether we want to perform calculations in one or the other of the two phases of the model, the symmetrical phase or the broken-symmetrical phase, whose existence and nature we discussed in section 1.1. In any case S_0 must satisfy the two essential conditions: it must be no more than quadratic on the fields and it must be stable, which means that it must correspond to a well-behaved theory of free fields, having therefore a lower bound.

The issue of stability must be examined carefully at this point. As we saw in section 1.1, in any continuum limit that does not approach the Gaussian point the where v_R is the expectation value of the field which, in the absence of any external sources breaking the discrete translation invariance of the lattice, as is our case here, should be constant, having the same values at all the sites. Since v_R is a constant it follows that the derivative term of the action remains unchanged. The polynomial terms which are quadratic and quartic on the fields, however, are transformed according to the relations

$$\begin{aligned}\varphi^2 &= \varphi'^2 + 2v_R\varphi' + v_R^2,\\ \varphi^4 &= \varphi'^4 + 4v_R\varphi'^3 + 6v_R^2\varphi'^2 + 4v_R^3\varphi' + v_R^4.\end{aligned}$$

We may neglect the constant terms, that do not depend on the field, since the exponentials of these terms are constant factors that appear both in the numerator and in the denominator of the ratio that defines the observables, thus cancelling off and not affecting in any way the statistical distribution of the model. Doing this we obtain for the complete action of the model

$$S = \sum_s \left[\frac{1}{2}\sum_\mu (\Delta_\mu\varphi')^2 + v_R\left(\alpha + \lambda v_R^2\right)\varphi' + \frac{\alpha + 3\lambda v_R^2}{2}\varphi'^2 + \lambda v_R\varphi'^3 + \frac{\lambda}{4}\varphi'^4 \right].$$

Since we know that α will always be strictly negative, we introduce now the parameter $\alpha_0 \geq 0$ and separate the action into a free part

$$S_0 = \frac{1}{2}\sum_\ell (\Delta_\ell\varphi')^2 + \frac{\alpha_0}{2}\sum_s \varphi'^2(s) \tag{1.2.4}$$

and an interaction part

$$\begin{aligned}S_I \;=\; \sum_s \Bigg[& v_R\left(\alpha + \lambda v_R^2\right)\varphi'(s) + \frac{\alpha - \alpha_0 + 3\lambda v_R^2}{2}\varphi'^2(s)\\ & + \lambda v_R\varphi'^3(s) + \frac{\lambda}{4}\varphi'^4(s)\Bigg].\end{aligned} \tag{1.2.5}$$

This is the form of the interaction term to be used in the broken-symmetrical phase of the model. We have therefore a completely well-defined scheme for trying to obtain approximations for the observables of the complete model in the vicinity of the Gaussian point, both in the symmetrical phase and in the broken-symmetrical phase. We must now perform in detail the calculation for some particular observables of the model, always keeping in mind that this is a very singular approximation scheme and that it may turn out that not everything will work as we might hope, in order to verify what we may learn about the structure of the model by means of the use of this technique.

Problems

(a) Show that this scheme would be sufficient to make the perturbative series well-behaved, with finite terms in the continuum limit, so long as, besides keeping at zero the average value of the quantity $\zeta - S_I$, we could also keep the *fluctuations* of this quantity at some finite average sizes around zero. Please note that we are *not* talking about the series being convergent, but only about its individual terms not diverging in the limit.

(b) Show that it is not possible to satisfy this condition in this model. In order to do this consider the calculation of $\langle(\zeta - S_I)^2\rangle_0$ under the condition that $\zeta = \langle S_I\rangle_0$, that is, calculate the quantity

$$\langle S_I^2\rangle_0 - \langle S_I\rangle_0^2$$

and show that it does not have a finite limit when $N \to \infty$.

(c) Repeat the first-order calculations presented in the text, using these ideas and the condition $\langle\zeta - S_I\rangle_0 = 0$ in order to determine ζ, thus showing that exactly the same results presented in the text are obtained in this context.

1.2.3. Perform explicitly the expansion of $f(\varepsilon)$ up to the order ε^2 and derive the form of the three terms that appear in equation (1.2.3).

1.3 Spontaneous Symmetry Breaking

Having developed in section 1.2 the ideas about the perturbative approximation for the observables of the $\lambda\varphi^4$ model, we will now discuss the calculation of some of the observables of the model to first order in ε which, in the cases to be examined here, is also known as the "one-loop" order[2]. The first thing that we will try to calculate will be the position of the critical curve near the Gaussian point. In order to to this we will examine the expectation value of the field,

$$v_R = \langle\varphi\rangle,$$

which functions as an order parameter for the phase transition that exists in the model. Of course, if we have in the model a non-vanishing external source j, then we should expect that v_R is also non-vanishing. The situation of spontaneous symmetry breaking is that in which we have $v_R \neq 0$ even when $j = 0$. Therefore, we will consider here the case $j = 0$ and try to verify whether or not it is possible to obtain solutions of the model with $v_R \neq 0$ in the limit in which $N \to \infty$. Observe that only in this limit of large lattices one can expect to obtain a situation of phase transition, with the existence of two distinct phases in the parameter plane of the model, separated by a phase-transition curve.

[2]These calculations were developed in collaboration with Dr. André Cavalcanti Rocha Martins.

$$\left\langle \varphi'(0) \sum_s \varphi'(s) \right\rangle_0 = \frac{1}{\alpha_0}. \tag{1.3.2}$$

For the other expectation value we obtain (problem 1.3.3)

$$\left\langle \varphi'(0) \sum_s \varphi'^3(s) \right\rangle_0 = \frac{3}{N^d \alpha_0} \sum_{\vec{k}} \frac{1}{\rho^2(\vec{k}) + \alpha_0}. \tag{1.3.3}$$

Observe how we avoided infrared problems in both cases, by the introduction of the non-zero parameter α_0. In these calculations all the strong divergences due to the behavior of S_I, which consist of terms proportional to N^d, cancel out. This fact corresponds, in the usual language of the traditional approach to the theory directly in the continuum, to the cancellation of the so-called "vacuum bubbles", and is a direct consequence of the fact that we are expanding a ratio of two functional integrals. We have therefore for our tadpole equation

$$\frac{(\alpha + \lambda v_R^2)}{\alpha_0} + \frac{3\lambda}{N^d \alpha_0} \sum_{\vec{k}} \frac{1}{\rho^2(\vec{k}) + \alpha_0} = 0.$$

We recognize now that the sum over the momenta is our already well-known quantity σ_0^2, the square of the width of the local distribution of the fields in the measure of S_0. We obtain therefore, substituting in terms of σ_0^2 and cancelling the factor of $1/\alpha_0$,

$$\lambda v_R^2 + \alpha + 3\lambda\sigma_0^2 = 0. \tag{1.3.4}$$

This equation gives us v_R for small values of $-\alpha$ and λ in the broken-symmetrical phase.

Let us consider here the issue of the dependence of this result on α_0. Observe that the result does not depend explicitly on α_0, but it may depend on this parameter through the squared width σ_0^2. For finite N the width does indeed depend on α_0 but, as was shows in the section in [11], in the continuum limit it does not depend on this parameter, so long as we make it go to zero sufficiently fast. More precisely, it suffices that we make $\alpha_0 = m_0^2 L^2/N^2$, for some finite m_0, for the limit to be completely independent of the value of m_0. The mass parameter m_0 could even be chosen to have the same value as the renormalized mass m_R of the model, but there is no need for this coming from this calculation, all we know up to now is that m_0 must be finite. Note that the need to choose α_0 dependent on N in a certain way in order to make the results independent of m_0 is already a first indication that the perturbative expansion is not completely well-behaved, since there should be no dependence at all on α_0.

Going back to the analysis of the critical behavior of the model, if we impose now that the *only* possible value for v_R be zero, we obtain from equation (1.3.4), by setting $v_R = 0$ in it, the equation of the critical curve, to wit

d	$\tan(\theta)$	θ (degrees)
3	$\simeq 1.3189$	$\simeq 52.83$
4	$\simeq 2.1515$	$\simeq 65.07$
5	$\simeq 2.8828$	$\simeq 70.87$

Table 1.3.1: Table of the slopes of the critical curves at the Gaussian point.

continuum limit. Hence, if the perturbative technique establishes at least a first-order approximation for the result of the complete model, then this result should be exact at the Gaussian point (problem 1.3.4). We will see later on that it is consistent with the results obtained by means of mean-field techniques and of stochastic simulations. In the case of the stochastic simulations realized so far, it has been verified that it is particularly difficult to execute them close to the Gaussian point, due to the fact that in that region the potential wells of the model become very shallow, which makes it more difficult to control the statistical errors. Therefore, up to now it has not been possible to do more than to confirm qualitatively this result with the stochastic simulations.

Using the asymptotic values of $\sigma_0^2(N)$ in the dimensions $d = 3$ to $d = 5$, presented in the section in [11], we obtain the results shown in table 1.3.1 for the slope, where θ is the smaller angle that the tangent line to the critical curve at the Gaussian point makes with the negative α semi-axis. It is interesting to observe that in the cases $d = 1$ and $d = 2$, since $\sigma_0(N)$ diverges, the slopes go to zero in the limit and the critical curve collapse onto the negative α semi-axis, where the model does not exist because this semi-axis is part of the unstable region. One might consider the interpretation that this is the perturbative way of verifying that the $\lambda\varphi^4$ model does not really exist as a quantum field theory for $d < 3$.

Our next objective is to calculate the propagator of the model, which we will do first in the symmetrical phase. We will denote the dimensionless two-point function of the complete model by

$$g(\vec{n}_1, \vec{n}_2) = \langle \varphi(\vec{n}_1)\varphi(\vec{n}_2)\rangle.$$

To order zero we simply have that $g(\vec{n}_1, \vec{n}_2) = g_0(\vec{n}_1, \vec{n}_2)$, where

$$g_0(\vec{n}_1, \vec{n}_2) = \langle \varphi(\vec{n}_1)\varphi(\vec{n}_2)\rangle_0,$$

so that in this order we have the results of the free theory, $\alpha_R = \alpha$ and a simple pole with its residue equal to one (apart from the normalization factor of $1/N^d$) in the region of imaginary momenta $\rho^2(\vec{k}) = -\alpha_R$. Note that this first-order result is not sufficient to allow us to take the continuum limit, because we know that α must become negative in the limit, while α_R cannot be negative. In the next-order approximation, using equation (1.2.3) up to first order, we will have a result that we shall denote by $g_1(\vec{n}_1, \vec{n}_2)$, with

$$\langle \varphi(\vec{n}_1)\varphi(\vec{n}_2)\rangle \approx g_1(\vec{n}_1, \vec{n}_2),$$

of this equation we recognize it as the expression in the equation of the critical curve that we calculated before, which shows that it does in fact go to zero in the $N \to \infty$ limit, so long as we take the limit in such a way that the parameters of the theory approach the critical curve. We therefore have here some perturbative evidence that the phase transition of the model is indeed of second order and we see once more why it is necessary to take the system to the critical situation in the continuum limit. In addition to this, as we already discussed before in this section, the expression in the right-hand side of the equation is positive in the symmetrical phase, showing that α_0 will be approaching zero by positive values and thus establishing the consistency of this choice for α_0. Observe that, with this choice for α_0, we may write the result for the propagator as

$$g_1(\vec{n}_1, \vec{n}_2) = \frac{1}{N^d} \sum_{\vec{k}} e^{\imath \frac{2\pi}{N} \vec{k} \cdot (\vec{n}_1 - \vec{n}_2)} \frac{1}{\rho^2(\vec{k}) + \alpha_R}, \tag{1.3.10}$$

that is, we get a propagator with form identical to that of the propagator of the free theory, with a renormalized mass m_R, where we see that $\alpha_R = \alpha_0$ and the renormalized mass is given by

$$m_R^2 = \lim_{N \to \infty} N^2 \alpha_R / L^2.$$

Formally, we may try to understand the expression $\alpha_R = \alpha + 3\lambda\sigma_0^2$ for the renormalized mass parameter α_R as the sum of a zero-order term α and a first-order term proportional to λ. However, in truth this is misleading, because we must recall that the parameter α is in fact negative in any continuum limit and hence that the first-order term cannot be considered as a small correction to the situation in the theory of the free field, in which α must be positive. We see that, in spite of the fact that we have developed this approximation technique in the lines of an expansion in a perturbative series, the resulting object has a character rather different from the expected.

We will see later on that the results of this process of approximation for the renormalized mass agree surprisingly well with the results of the stochastic simulations. In particular, note that the result indicates a unit residue for the pole of the propagator, exactly as in the free theory, α_R being the only non-trivial parameter that appears. This unit residue is also found in all the stochastic simulations, within the statistical errors. Judging by the form of the propagator, one would say that the spectrum of the theory seems to be that of free particles with mass m_R. One might consider interpreting this as an indirect perturbative indication related to the underlying triviality of the model. At least, the result for the residue is compatible with it.

It is interesting to try to understand in clearer physical terms the nature of the approximation technique that we have developed. The crucial point for the success of the technique is the choice of α_0, which ends up being equivalent to a preliminary implicit choice $\alpha_0 = \alpha_R$, to be resolved after the end of the calculation, a possibility that was suggested in section 1.2. From the very beginning we are trying to which is a positive quantity in this phase. Once again the expression in the equation of the critical curve appears, showing once more that α_R will go to zero when we approach this curve in the continuum limit, this time by the other side, from the broken-symmetrical phase. The factor of 2 that appears in this result confirms once more our heuristic expectations and, as we will see later on, it also matches with surprising precision the numerical results in this phase.

Observe that, since σ_0^2 is a function of α_R, both this result and the result for the symmetrical phase are not explicit solutions for α_R but rather equations that determine α_R in an implicit way,

$$\alpha_R - \alpha - 3\lambda\sigma_0^2(\alpha_R) = 0$$

in the symmetrical phase and

$$\alpha_R + 2\alpha + 6\lambda\sigma_0^2(\alpha_R) = 0$$

in the broken-symmetrical phase, where

$$\sigma_0^2(\alpha_R) = \frac{1}{N^d}\sum_{\vec{k}} \frac{1}{\rho^2(\vec{k}) + \alpha_R}.$$

It is not difficult to determine the existence, the number and the character of the solutions of these equations, if one separates from the sum in σ_0^2 the term containing the zero mode, and to find these solutions on finite lattices by numerical means (problems 1.3.8, 1.3.9, 1.3.10 and 1.3.11). We show in figure 1.3.1 a graph with some of the numerical solutions, illustrating their behavior for lattices of increasing size. In this graph, instead of the usual Cartesian coordinates α and λ in the parameter plane of the model, we use polar coordinates centered at the Gaussian point, with the radius r given by $\sqrt{\alpha^2 + \lambda^2}$, and the angle θ defined as the angle between the radius vector (α, λ) and the negative α semi-axis.

The equation for the symmetrical phase has two solutions, but only one of them is positive and hence physically meaningful. While the positive solution remains finite and non-vanishing in the $N \to \infty$ limit, the negative solution becomes identically zero in the limit. Note that this equation has solutions over the whole parameter plane and not only in the symmetrical phase. The curves corresponding to this solution are the ones with their maximums at the right in figure 1.3.1. Of course this solution only has meaning in the symmetrical phase, but since this is not a well-defined concept on finite lattices, in order to determine the range of validity of the solution on finite lattices, from the point of view of perturbation theory, we must first discuss the solutions in the broken-symmetrical phase. The equation for the broken-symmetrical phase has two positive solutions, a small one and a large one, but only for certain values of parameters, thus defining a perturbative broken-symmetrical phase even on finite lattices. In the complementary region of the parameter plane, which we might call the perturbative symmetrical phase, the equation has no real solutions. On each finite lattice the position of the curve separating these two regions approaches the free theory on finite lattices, with this new value for the dimensionless mass parameter. As one can see in figure 1.3.1 in this case the large solution does indeed approach $\alpha_R = -2\alpha$ as expected, where $-2\alpha = 2r = 0.2$. Just as in the case $\theta = 180^{\rm o}$ the solution seems to be exact in this case, because for θ approaching $0^{\rm o}$ we have a vanishing λ and the distribution tends to become purely Gaussian, so that the Gaussian approximation tends to become a perfect one. It is therefore to the two large solutions that we should attribute physical meaning, using the broken-symmetrical solution where it exists and using the symmetrical solution where the broken-symmetrical solution does not exist, and therefore truncating in this way the symmetrical solution. Note that there is a discontinuity between the two solutions at the transition point, but this discontinuity vanishes in the $N \to \infty$ limit. In this limit the edges of the two curves approach the value $\alpha_R = 0$ at the critical line.

In the $N \to \infty$ limit σ_0^2 becomes independent of α_R, the critical transition becomes completely well-defined and the relations between α_R and the parameters (α, λ) become linear, in either phase. If we define $\sigma_\infty^2 = \sigma_0^2(N \to \infty)$, then we have in the symmetrical phase

$$\alpha_R = \alpha + 3\sigma_\infty^2\lambda,$$

and in the broken-symmetrical phase

$$\alpha_R = -2\alpha - 6\sigma_\infty^2\lambda,$$

which are equations of planes over the (α, λ) parameter plane. The three-dimensional graph of $\alpha_R(\alpha, \lambda)$ over the parameter plane is composed of pieces of two planes that intersect within the (α, λ) plane at the critical line. The first one is a piece of the plane defined by the critical line and the line $\alpha_R = \alpha$ within the vertical (α, α_R) plane, the second one is a piece of the plane defined by the critical line and the line $\alpha_R = -2\alpha$ in that same vertical (α, α_R) plane. The relevant part of the first plane is the part that stands over the symmetrical phase, the relevant part of the second plane is the part that stands over the broken-symmetrical phase.

Note that both on finite lattices and in the $N \to \infty$ limit the perturbative solution for the expectation value v_R of the field is proportional to the value of the renormalized mass in the broken-symmetrical phase, that is, we have

$$v_R^2 = \frac{\alpha_R}{2\lambda}.$$

Due to the extra dependence on λ, we can have independent values of the two dimensionless renormalized quantities, v_R and α_R. Whether or not the same is true for the corresponding dimensionfull quantities V_R and m_R depends on the dimension d of space-time, because the dimensions of the field and therefore of V_R depend on it (problem 1.3.12).

The final conclusion of this effort is that the perturbative technique of Gaussian approximation allows us to calculate in a useful way the observables related to the aspects of propagation of particles and to the aspects of spontaneous symmetry

$$\alpha_R - \alpha - 3\lambda\sigma_0^2(\alpha_R) = 0,$$

were we recall that

$$\sigma_0^2(\alpha_R) = \frac{1}{N^d}\sum_{\vec{k}}\frac{1}{\rho^2(\vec{k}) + \alpha_R},$$

has a single positive solution for each pair of values (α, λ) in the stable region of the parameter plane of the model. In order to do this, remember that the parameter α_R has to be positive or zero and consider the behavior of the left side of the equation when $\alpha_R \to 0$ and when $\alpha_R \to \infty$. Remember that the sum that defines the quantity σ_0^2 includes the zero mode and write it as

$$\sigma_0^2 = \sigma_0'^2 + \frac{1}{N^d\alpha_R},$$

where $\sigma_0'^2$ has a finite limit for $\alpha_R \to 0$. Show also that there is a second solution, which is negative (and hence destitute of any physical meaning) and which becomes identically zero in the $N \to \infty$ limit.

1.3.9. ($\star$) For given α, λ and N, write a program to solve numerically the equation

$$\alpha_R = \alpha + 3\lambda\sigma_0^2(\alpha_R)$$

for $\alpha_R(\alpha, \lambda)$.

1.3.10. Show that the perturbative equation which determines the renormalized mass parameter α_R in the broken-symmetrical phase, as a function of α and λ,

$$\alpha_R + 2\alpha + 6\lambda\sigma_0^2(\alpha_R) = 0,$$

has two different real and positive solutions for some pairs of values (α, λ) in the stable region of the parameter plane of the model, and no real solutions for other pairs of values. Use the same ideas and techniques that were suggested in problem 1.3.8. Show that the condition on α and λ for the existence of solutions can be written in an implicit way, which depends on α_R on finite lattices, as

$$(\alpha + 3\lambda\sigma_0'^2)^2 \geq \frac{6\lambda}{N^d},$$

and interpret the meaning of this condition on the continuum limit. Show that when the equality holds in the condition above the renormalized mass parameter is given by $\alpha_R = \sqrt{6\lambda/N^d}$, and hence that it goes to zero at the critical curve in the continuum limit. Show also that the smaller of the two solutions becomes identically zero in the $N \to \infty$ limit.

1.3.16. Show, using the first-order perturbative results obtained in the text, that in the complete model the observable σ_1 given by $\sigma_1^2 = \langle\varphi^2\rangle$ in the symmetrical phase and by $\sigma_1^2 = \langle\varphi'^2\rangle$ in the broken-symmetrical phase, is equal, in either case, to the observable σ_0^2 of the free theory. Observe that this shows that the field fluctuates in a similar way in either model, undergoing fluctuations with the same typical size.

1.3.17. Show that it is possible to take continuum limits to the Gaussian point over the positive λ semi-axis, that is, keeping $\alpha = 0$ constant during the limit. Determine how to take this type of limit so that the renormalized mass is finite and non-zero, that is, discover how $\lambda(N)$ must be so that m_R has a finite and non-zero limit. This type of limit, which produces a non-zero renormalized mass without involving any parameters with dimensions of mass from the corresponding classical theory, is related to what has been conventionally called the phenomenon of "dimensional transmutation".

Chapter 2

The Sigma-Model Limit

In this chapter we will discuss the infinite-coupling limit of the polynomial models, which is, in a way, the antithesis of the perturbative approach. We will see that in the $\lambda \to \infty$ limit there is an *exact* representation of the polynomial models, which is another class of non-linear models of scalar fields, known as the *sigma models*. As we will see in the simple case we will deal with here, in this limit the quantum $\lambda\varphi^4$ models can be identified with a simple example of these sigma models. In the one-component models that we are examining here these sigma models reduce to the well-known Ising models. Note that we are dealing here with an equivalence between the *quantum* versions of the models, not between the corresponding classical field theories, which are quite different from one another.

2.1 Derivation of the Ising Model

In this section we will take the $\lambda \to \infty$ limit of the $\lambda\varphi^4$ model. This can be done for more general models than the one-component $\lambda\varphi^4$ model we are examining here, namely the multi-component $\lambda\varphi^4$ models which are invariant by the $SO(\mathfrak{N})$ groups of transformations. In general the $\lambda \to \infty$ limit of these models will take us to the corresponding $SO(\mathfrak{N})$-invariant sigma models. In our case here, however, we will deal only with the one-component $\lambda\varphi^4$, which is invariant by the sign reflections. The simple discrete set of transformations given by the identity and the reflection also forms a group of transformations, a discrete group which is known by either $O(1)$ or $\mathbb{Z}_2$. In this case the corresponding sigma model is simply the Ising model which was mentioned in [13]. In this way we will establish that we can use the Ising model as a direct representation of the infinite-coupling limit of the $O(1)$-symmetrical $\lambda\varphi^4$ model.

The Ising model can be obtained from the $\lambda\varphi^4$ polynomial model in the limit in which the coupling parameter λ tends to positive infinity over negative-slope straight lines in the parameter plane of the $\lambda\varphi^4$ model. These lines must exist only for $\lambda > 0$, because otherwise they would cross the region where the model is unstable. Also, their slopes must be strictly negative (not zero) and finite, which rules out horizontal and vertical lines in the parameter plane. Since the slopes must

The functional integrals that appear in the definition of the observables of the quantum theory may now be written as

$$\int [\mathbf{d}\varphi]\; e^{-S[\varphi]} = \int [\mathbf{d}\varphi]\; e^{-S_V[\varphi]}\; e^{-S_K[\varphi]},$$

so that, including the exponential of S_V in the measure of the distribution of the model, we may write this distribution as

$$[\mathbf{d}\varphi]\; e^{-S[\varphi]} = \left[\mathbf{d}\varphi\; e^{-V(\varphi)}\right] e^{-S_K[\varphi]}.$$

Since we may multiply this quantity by any constant independent of the fields without changing the observables, we may write the measure as

$$[\mathbf{d}\varphi\; \varrho(\alpha, \lambda, \varphi)],$$

where $\varrho(\alpha, \lambda, \varphi)$ is the local part of the distribution, which we thus include in the measure, normalized as

$$\varrho(\alpha, \lambda, \varphi) = \frac{e^{-V(\varphi)}}{\displaystyle\int_0^\infty \mathrm{d}\varphi\; e^{-V(\varphi)}}. \tag{2.1.1}$$

Observe that the integration runs only over the positive values of φ and hence can be interpreted as an integration over the absolute value of the field. With this the following normalization condition for ϱ holds,

$$\int_0^\infty \mathrm{d}\varphi\; \varrho(\alpha, \lambda, \varphi) = 1.$$

The integration over the absolute value of the field, when generalized to the more complex models having $SO(\mathfrak{N})$ symmetry, corresponds to the separation of the integration variables into a radial part and an angular part. In this way the arguments being presented here can be easily generalized to that case. In general the radial variable will be the modulus of the $\mathfrak{N}$-dimensional field vector in the internal symmetry space, which in our case here reduces to the absolute value of φ. Also, in our discrete $O(1)$-symmetrical case the integral over the angular part reduces to a sum over only two points, corresponding to rotations by each one of the two angles 0 and π, and hence to the two possible signs, $\cos(0) = 1$ and $\cos(\pi) = -1$. Therefore, in our present case the integral over all the values of φ at each site is being decomposed in this way into an integral over the positive values of φ and a sum over the two possible signs. Since both $\varrho(\alpha, \lambda, \varphi)$ and $V(\varphi)$ are even functions of φ, they are in fact functions only of the absolute value of φ, and independent of its sign, so it is not necessary to make their dependence on the absolute value explicit. Given all this, we may write the functional integrals as

$$\int [\mathbf{d}\varphi\; \varrho(\alpha, \lambda, \varphi)]\; e^{-S_K[\varphi]} = \sum [\psi = \pm 1] \int [\mathbf{d}|\varphi|\; \varrho(\alpha, \lambda, \varphi)]\; e^{-S_K[\psi\varphi]},$$

where the remaining part of the measure can be written explicitly as

$$[\mathbf{d}|\varphi|\ \delta(\varphi^2-\beta)] = \prod_s \mathrm{d}|\varphi(s)|\ \delta[\varphi^2(s)-\beta],$$

where the Dirac delta functions imply a condition of constraint on the fields, $\varphi^2 = \beta$, or $\varphi = \sqrt{\beta}$, since the sign of φ is being considered explicitly and hence φ is positive. We may now use the Dirac delta functions to perform all the integrations over $|\varphi|$, thus obtaining for the expectation values

$$\langle\mathcal{O}\rangle_N = \frac{\sum[\psi=\pm 1]\ \mathcal{O}[\sqrt{\beta}\psi]\ e^{-S_K[\sqrt{\beta}\psi]}}{\sum[\psi=\pm 1]\ e^{-S_K[\sqrt{\beta}\psi]}}.$$

We may now examine the form of the action S_K under these conditions, in order to simplify it and exhibit it in a more familiar form. In terms of the new variables ψ and the parameter β this action can be written as

$$S_K[\psi] = \frac{\beta}{2}\sum_\ell(\Delta_\ell\psi)^2,$$

where the new field variables $\psi = \varphi/\sqrt{\beta}$ satisfy the constraint $\psi^2 = 1$, since they are just signs, and the parameter β appears now multiplying the action, just like the parameter $\beta = 1/(kT)$ of statistical mechanics. Although we may want to think of β as the inverse of a fictitious temperature, in order to guide our intuition about the behavior of the models, we should remember that our β is, in truth, related to the mass parameter α and the coupling constant λ, and *not* to any truly physical temperature related to the dynamical system we are studying. The phenomenon that something relating to a coupling constant appears multiplying the action due to a scaling redefinition of the fields is typical of the gauge theories, as we may see in future volumes.

Note that, although $S_K[\psi]$ is a purely quadratic functional of ψ, the model is *not* the free theory, due to the fact that the field ψ satisfies an equation of constraint, and is not, therefore, a free real variable. This is a situation in which the non-linearities, instead of being introduced by a term in the action, are introduced instead by the measure of the functional integral, which is where the constraint is implemented in the quantum theory. We may now perform one more transformation of the action of the model, with the intention of showing in a clearer way its relation with the Ising model of statistical mechanics. If we expand the squares of the derivatives contained in the action, we get

$$(\Delta_\ell\psi)^2 = \psi^2(\ell_-) - 2\psi(\ell_-)\psi(\ell_+) + \psi^2(\ell_+),$$

where $\psi(\ell_-)$ and $\psi(\ell_+)$ are the fields at the two ends of the link ℓ. Using now the equation of constraint $\psi^2 = 1$ we see that the two terms containing the squares are constant, independent of the fields, which means that they can be neglected in the

the potential term S_V. We may therefore introduce external sources into the resulting Ising model exactly as we would do in the original polynomial model, and the whole functional generator formalism is made available for the analysis of the Ising model, without any change. Hence the Ising model is in fact an exact and complete direct representation of the infinite-coupling limit of the corresponding $\lambda\varphi^4$ polynomial model. This relationship can be generalized to the multicomponent $SO(\mathfrak{N})$-invariant polynomial models and the corresponding sigma models. It can also be generalized to models with larger powers of the fields. It is possible to show (problem 2.1.5) that the Ising model can also be obtained as the $\lambda \to \infty$ limit of the $\lambda\varphi^{2p}$ polynomial models, with $p = 3, 4, 5, \ldots$, in a way which is completely analogous to the $p = 2$ case that we examined here.

Another aspect which we must examine here is the one relating to the superposition process involved in the definition of the block variables. Once again this is inherited by the Ising model from the polynomial model, so we still have a simple linear superposition of the fields ψ at the various sites within the block. Note that although these fundamental fields satisfy the constraint $\psi^2 = 1$, the same will not be true for the block variables. If we consider the process of linear superposition of the fields within a block, in order to define a block variable, it is evident (problem 2.1.6) that the sum and the average of a collection of signs $\psi = \pm 1$ will not itself have unit absolute value. If the fields are distributed is a very random way, without any appreciable alignment, their average will tend to have an absolute value much smaller than 1. Only in the opposite case, when the fields are highly aligned, the absolute value of the average will tend to 1. The absolute value of the sum may have any value, either larger or smaller than 1.

We see therefore that in general the introduction of external sources will cause the block variables, which are the variables in terms of which we should interpret the theory, to deform to any value, without respecting an equation of constraint. In fact, they will behave much like the corresponding variables of the polynomial models. In short, we see that the constraint that appears for the fundamental field in the large-coupling limit does not survive the block-variable superposition process and that the Ising model we get in the limit is not fundamentally different from the $\lambda\varphi^4$ polynomial model it derives from. Hence we confirm that the Ising models are not to be seen as a completely different class of models, but as a way to examine directly the behavior of the polynomial models in the $\lambda \to \infty$ limit. Since, as was mentioned before, the Ising models can also be obtained as the limits of the $\lambda\varphi^{2p}$ polynomial models for any $p \geq 2$, they become a tool for the examination of the large-coupling limit of a large class of models.

We will finish this section with some comments about the critical behavior of the models. As we shall see later, approximate calculations based on the mean-field technique show that the Ising models have well-defined critical points β_c for dimensions $d \geq 3$. Since they are the $\lambda \to \infty$ limits of the polynomial models, we see that the perturbative Gaussian approximation also predicts well-defined critical points for the Ising models, although they are infinitely distant from the Gaussian point. In fact, the two predictions do not differ very much from each other, and are

2.1.5. Repeat in a qualitative way the derivation presented in the text for the case of the $\lambda\varphi^4$ models, in order to show that the Ising models are obtained, in limits in which $\alpha \rightarrow -\infty$ and $\lambda \rightarrow \infty$, with constant $-\alpha/\lambda$, from the corresponding $\lambda\varphi^{2p}$ models, for any $p \geq 3$.

2.1.6. Show that the average of the field ψ over any number of sites has an absolute value smaller than or equal to 1. Show also that the absolute value of the sum of ψ over a block of sites may have any positive value in the continuum limit.

2.2 The Mean Field Method

In this section we will introduce the mean-field approximation method. This is an approximation process which is not perturbative, not being based on the Gaussian model. It is a traditional approximation method of statistical mechanics, which is also useful for obtaining non-perturbative approximations in quantum field theory. It is easy to use it to obtain approximations for local quantities defined on lattice sites, such as, for example the expectation value of the field. It is a well-known fact that the results of the method tend to improve with the increasing dimension of the space where the models are defined. Usually the results are reasonably good in three dimensions and even better in four dimensions, while the method often fails completely in one and two dimensions. There are even some speculations that, for some quantities, the method becomes exact in sufficiently large dimensions. In its usual formulation this method does not establish a series of successive approximations, but rather a single approximation, making it difficult to evaluate the errors involved in this approximation. We will introduce here an extension of the method, that improves the situation and allows us to understand its bad performance for small dimensions.

The formulation we will present is specifically for systems defined on the Euclidean lattice, with interactions only between next neighbors. As we will see, it is intimately related to stochastic simulations of the systems on finite lattices in which one uses a certain type of fixed boundary conditions, which we denominate *self-consistent* boundary conditions. These structures on finite lattices constitute an extension of the usual mean-field method and, unlike the usual method, they give us a whole series of successive approximations. The first of them will be the usual approximation, while the subsequent approximations converge to the exact solution of the models within a finite box with fixed boundary conditions, as the lattice spacing is decreased. This extension of the method is similar, but not identical, to other well-known extensions of the traditional method in statistical mechanics, such as the Oguchi method and the Bethe, Peierls, Weiss and Kikuchi [29] method. This approximation method can be used both in the polynomial models and in the sigma models, we will assume only that we have an $O(1)$ model of scalar fields, on a lattice of dimension d, with the usual forms of the action. As we saw in sections 1.1 and 2.1, the action of any of these models can always be separated in two parts, a strictly local one and one involving only interactions between next neighbors, which

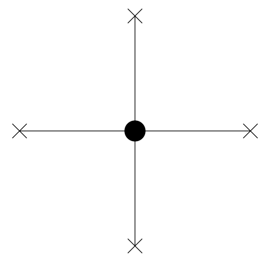

Figure 2.2.1: Lattice with $N = 1$ and fixed boundary conditions, related to the usual mean-field approximation.

at any of the neighboring sites, assuming that they are all equivalent by discrete translation invariance. This means that we are using for the external field the value $j_{\rm MF} = 2d\langle\varphi\rangle$. The calculation of this average value is made, in the context of this method, a-posteriori and in a self-consistent way: one calculates the average value at the active site located at $\vec{n}$ and imposes that this value be equal to the average value at the neighboring sites.

In this way we replace the detailed interactions between the field at each site and the fields at the neighboring sites by an interaction at each site with a background field, which does not fluctuate, thus rendering the problem mathematically more tractable. It is interesting to observe that the spirit of this approximation is somewhat different from the spirit of the usual approximation in statistical mechanics, in which we consider the interaction of the dynamical variables with an external mean field, ignoring completely that the interactions are established through the links of the lattice. Observe that, in our case here, the dimension of the lattice appears explicitly in the approximation. However, this distinction will only be really relevant when we consider the extension of the method to clusters of sites. For the case discussed so far, in which only the field at a single site is kept active, that is, undergoing statistical fluctuations, the two methods are identical. They are known in statistical mechanics as the *constant coupling method*, which was developed by Yvon, Nakamura, Kasteleijn, Van Kranendonk, Kikuchi and Callen [29].

This replacement of the interactions between next neighbors by an interaction with a non-dynamical field that undergoes no fluctuations is clearly a very radical change and it is rather surprising that it can produce good results, even if only for some observables. In particular, since the dynamical fields at each site interact only with the constant background field and no longer with each other, it is clear that the fields at the various sites will become completely uncorrelated from each other in this approximation, so that the calculation of correlation functions is out of the question. There is, however, an alternative interpretation of the method, which will allow us to extend it to clusters of active sites and hence to recover the correlations among sites. This alternative interpretation, which changes nothing in the mathematics involved, is that the fields at all the sites are frozen at their average values, except for a single arbitrarily chosen site, which remains active. Since in the traditional mean-field method all the sites are equivalent and uncorrelated from

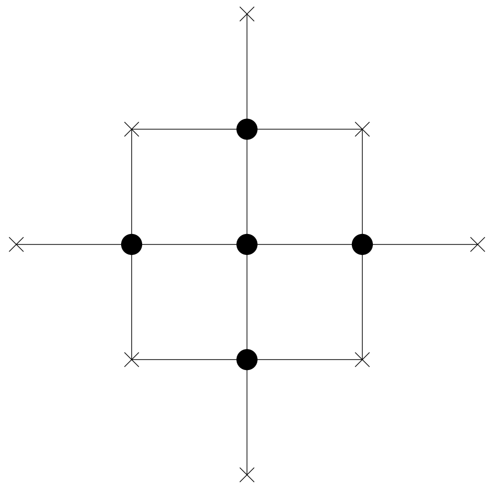

Figure 2.2.3: Modified lattice used in the Bethe-Peierls cluster method.

This extension of the mean-field method is similar to the method of clusters of Bethe, Peierls, Weiss and Kikuchi in statistical mechanics, in which groups of connected sites are considered. The first cluster considered in this method is the Bethe-Peierls cluster, which is shaped like a diamond as shown, within the context of our lattice representation, by the diagram in figure 2.2.3, including the border sites, which are not part of the original cluster method. Larger clusters with formats similar to this one may also be considered and used for analytical calculations. However, the amount of analytical work involved is usually quite large, to achieve only modest gains in the quality of the results obtained.

In the general case our extension is not identical to the traditional cluster method, because in our case the interaction of the cluster with the mean field is established only through the border, not by means of an external mean field that acts also on the internal sites of the cluster, which have no direct contact with the border. This kind of internal site appears in the Bethe-Peierls cluster and also in the cubical clusters starting from $N = 3$, as illustrated in the diagram in figure 2.2.4. The two methods also differ regarding the type of self-consistency condition which is imposed. In the case of the Bethe-Peierls cluster, rather than adjusting the external mean field so that it becomes consistent with the average value of the fields at the active sites, what is done is to adjust it so that the average of the field at the central site is identical to the average of the field at the other $2d$ sites of the cluster, which are all equivalent to each other due to the symmetry of the cluster. Hence, what one actually imposes in this case is that the normal derivative of the average value of the field vanish at the border.

It is clear that the $N \to \infty$ limit of our sequence of cubical clusters produces exactly the continuum limit within a finite cubical box with a certain type of self-consistent boundary conditions. Let us now discuss, in greater detail, the self-consistency condition to be imposed on the border sites. The usual $N = 1$ mean-field approximation is sufficient for the calculation of strictly local quantities, defined at value at the border, the value at the border is modified so as to coincide with the value measured in the interior. This is done many times at regular intervals, so that eventually the adjustment of the value at the border becomes negligible and the border fields stay at the desired value. From this moment on one can start measuring whatever observables one may be interested in. For lattices larger than that of the case $N = 1$ a similar mechanism may be used, but this time there are several possible variations of the procedure. For example, we may measure and feed back to the border the average value calculated for the spacial average of the fields over all the internal sites or, alternatively, we may use a spacial average over only the internal sites which are in direct contact with the border, thus implementing a self-consistency condition in the spirit of the Bethe-Peierls condition, involving the normal derivative at the border.

One of the most interesting properties of these systems with fixed but self-consistent boundary conditions relates to their characteristics of critical behavior. Usually we build the models and their corresponding stochastic simulations on finite lattices with periodical boundary conditions[1]. These systems suffer from the inconvenience that there is no true critical behavior on finite lattices of this kind, that is, in systems with a finite number of degrees of freedom and no external boundary [30]. For example, if we calculate by means of stochastic simulations on finite lattices the expectation value v_R of the field in the Ising models, a quantity which is analogous to the magnetization, as a function of the inverse temperature β, we typically obtain curves $v_R(\beta)$ that are continuous, differentiable and monotonically increasing. There are no sharp transitions except in the $N \to \infty$ limit, which makes it considerably more difficult to extract from these simulations the critical values of the parameters of the models by means of extrapolations of the finite-lattice results to the $N \to \infty$ limit.

In contrast to this, the self-consistent lattice systems display sharp transitions and complete critical behavior even on finite lattices. In the case $N = 1$ the fact that the curve of the magnetization displays a sharp transition at a certain critical value of β, a point where it is not differentiable, can be verified analytically. In numerical simulations the sharpness of the transitions is limited, of course, by the technical and numerical limitations of the computer simulations but, with increasing expenditure of computer resources, the transitions can, at least in principle, be made as sharp as one desires for any given N, quite unlike the case of periodical simulations. These self-consistent simulations are, therefore, potentially better for the calculation of critical quantities. With simulations for larger values of N we can not only improve the calculation of local quantities such as the expectation value of the field, we can also calculate significant approximations for non-local quantities, such as the correlation functions for the theory defined within a finite box. This is, therefore, a very useful extension of the mean-field method. Clearly, there is a numerical price

[1]For a more complete understanding of the rest of this section it is useful, although not completely essential, to have some knowledge of the simulation techniques and some experience with stochastic simulations. This subject will be examined in detail in another volume of this series of books.

time dimension. In this way a certain $\beta_c(N)$ is defined on each finite lattice, which is the critical point for that lattice size, assuming that we use as an example a sigma or Ising model. In the cases in which there is a phase transition in the $N \to \infty$ limit $\beta_c(N)$ tends to a finite value β_c^* when $N \to \infty$. In the cases in which there is no phase transition in the limit what happens is that $\beta_c(N)$ increases without limit when N goes to infinity. Since in all cases we have the non-oriented phase for $\beta < \beta_c$, in these cases this random phase is the only one that remains in the $N \to \infty$ limit.

Thus we see that, if there is a phase transition for some well-defined finite β_c^* in the $N \to \infty$ limit, then the $N = 1$ approximation, which also displays a well-defined phase transition, will not be qualitatively different from the limit, although it may be quantitatively quite different, thus giving rise to an approximation that is interpreted as successful. In contrast to this, in the case in which there is no phase transition in the limit $N \to \infty$ the $N = 1$ approximation, because it always displays a well-defined phase transition, becomes qualitatively different from the limit and is thus interpreted as a complete failure. Observe that the $N = 1$ approximation fails precisely in the cases in which the system does *not* display critical behavior in the limit of large values of N, cases which are not, therefore, of much interest for us.

The situation regarding the realization of the theorems is not very different for periodical boundary conditions, except for the fact that in this case there are no well-defined critical points on finite lattices. However, in order to tackle this case, we must first dispel a common misconception regarding finite lattice systems with periodical boundary conditions. Although it is true that if one measures the expectation value of the field v_R in such systems one gets zero within errors, this does not really mean that the single phase existing on finite lattices is the symmetrical phase. The reason why one gets zero for v_R in these circumstances is not that the field configurations are typically non-oriented, but rather that the average value us washed out by the wandering of the direction of alignment. The best way to describe what happens is to say that the system is always in the broken-symmetrical phase on finite lattices, and that it only becomes symmetrical in a certain region of its parameter space in the $N \to \infty$ limit.

One can verify this fact in at least three ways, which we now describe shortly. First, one can include in the action a constant external action j and verify that the resulting value of v_R does not vanish in the limit in which $j \to 0$. Second, one can consider looking at the expectation value of the average of the field over the lattice, which is just the zero mode, the zero-momentum Fourier transform $\widetilde{\varphi}(\vec{0})$ which is like the magnetization in statistical mechanics; if one measures both its expectation value and the expectation value of its square, one gets zero for the first but not for the second, meaning that the field configurations are typically oriented, and that the direction of this orientation drifts. Third, one can eliminate the drift on finite lattices by hand by freezing the zero mode of the field in an arbitrarily chosen direction, and then verifying that one gets explicitly a non-vanishing average magnetization; this changes nothing in the $N \to \infty$ limit, since in this limit the drift is frozen in any case.

is the order parameter of these models. It can be defined as the expectation value v_R of the field, which will be pointing predominantly in the direction that we choose arbitrarily for the symmetry breaking. This is the direction in which we will keep oriented the fields at the border. We will calculate v_R in the mean-field approximation as a function of the parameter β and of the dimension d of space-time. As we saw before in section 2.2, the parameter β can be understood as the inverse of a fictitious temperature. For a given value of d, the critical point β_c is the value of the inverse temperature for which the magnetization $v_R(d, \beta)$ ceases to be zero, when we "cool" the system and therefore increase β. For the calculations it is convenient to use the dimensionless field ψ that was introduced in section 2.2, so that for the purposes of this section we will use, rather than $v_R = \langle\varphi\rangle$, the quantity $\nu_R = \langle\psi\rangle$, where $v_R = \sqrt{\beta}\nu_R$. It is ν_R rather than v_R that is more closely related to the standard definition of the magnetization of statistical mechanics. As was derived in that section, the action can be written in terms of ψ as

$$S[\psi] = -\beta \sum_{\ell} \psi(\ell_-)\psi(\ell_+),$$

while the constraint is written as

$$\psi^2 = 1,$$

where the field ψ is, therefore, either $+1$ or -1. We now freeze all the sites except one at the value $\psi = \nu_R$, arbitrarily choosing the positive direction as the direction of orientation of the fields. Therefore in our $N = 1$ mean-field approximation the relation $\nu_R = \langle\psi\rangle$ can be written as

$$\nu_R = \frac{\sum[\psi = \pm 1]\ \psi\ e^{2d\,\beta\nu_R\psi}}{\sum[\psi = \pm 1]\ e^{2d\,\beta\nu_R\psi}}.$$

Since the sum is over a single site, in this simple $O(1)$ case we can immediately write the result as

$$\nu_R = \frac{\sinh(2d\,\beta\nu_R)}{\cosh(2d\,\beta\nu_R)}, \tag{2.3.1}$$

thus obtaining an algebraic equation that in principle gives us the complete solution for ν_R in the $N = 1$ system. This equation can be solved numerically in order to provide us with graphs of ν_R as a function of β (problem 2.3.2). It can also be used to determine the value of β at which $\nu_R = 0$ becomes the only possible solution (problem 2.3.1), by the use of series expansions. However, the simplest way to obtain the critical points is to simply expand its right-hand side around $\nu_R = 0$. This is sufficient to determine the critical points because the phase transition is of second order in these models and, therefore, we have that $\nu_R \approx 0$ for $\beta \approx \beta_c$. Expanding and keeping only up to first-order terms (problem 2.3.3) we get

d	$\mathfrak{N}$	β_c	
		periodical	mean-field
1	1	∞	0.500
2	1	0.4406868	0.250
3	1	0.22165(4)	$0.16\bar{6}$
4	1	0.15 ± 0.02	0.125

Table 2.3.1: Critical points of the Ising models in dimensions from $d = 1$ to $d = 4$.

the mean-field result deviates about 17% from the numerical results, while for $d = 3$ it deviates by about 25%.

Calculating the magnetization in an infinitesimal neighborhood of β_c one can also obtain the mean-field approximation for the corresponding critical exponent. The definition of this critical exponent is as follows. If we have, close to the critical point in the broken-symmetrical phase,

$$\nu_R \approx C\,(\beta - \beta_c)^{\gamma} \tag{2.3.3}$$

for some non-vanishing constant C, then γ is the critical exponent of ν_R. Once more, this can be done using the complete solution in terms of the hyperbolic functions (problem 2.3.4), but the simplest way to obtain the result is to expand equation (2.3.1) to third order in ν_R (problem 2.3.3), thus obtaining

$$\nu_R = 2d\,\beta\nu_R - \frac{8}{3}d^3\beta^3\nu_R^3.$$

Once more a factor of ν_R cancels out and we obtain

$$1 = 2d\,\beta\left[1 - \frac{4}{3}d^2\beta^2\nu_R^2\right].$$

Remembering that $\beta_c = 1/(2d)$ and considering that we are in the vicinity of the critical point, with $\beta \approx \beta_c$, we may write this as

$$\nu_R = \sqrt{3}\,\frac{\beta_c}{\beta}\sqrt{\frac{\beta - \beta_c}{\beta}} \approx \sqrt{3}\sqrt{\frac{\beta - \beta_c}{\beta_c}} = C(\beta - \beta_c)^{1/2}. \tag{2.3.4}$$

This shows that the mean-field approximation for the critical exponent is $\gamma = 1/2$, and determines the value of the constant $C = \sqrt{3/\beta_c}$. This value for the critical exponent of the magnetization is characteristic of the mean-field calculations with $N = 1$.

Another mean-field calculation of interest is that of the critical curves of the $\lambda\varphi^4$ polynomial models, which are given by equation of the type $\lambda = f(\alpha)$. In this case we will use the usual dimensionless field φ as our variable. If we define the magnetization for this case as $v_R = \langle\varphi\rangle$ we may write, in the $N = 1$ mean-field approximation, Although we are not able to solve equation (2.3.6) analytically in order to write the equation $\lambda = f(\alpha)$ of the critical curve in explicit form, it is possible to solve the asymptotic form of the equation, for large values of λ (problem 2.3.8). Doing this we discover that in this limit the critical curve is asymptotic to the straight line defined by the equation

$$\lambda(\alpha) = -\frac{1}{\beta_c}(\alpha + 2d),$$

where $\beta_c = 1/(2d)$ are the critical points of the corresponding Ising models, so that the asymptote cuts the λ axis at the point $-4d^2$ and forms with the negative α semi-axis an angle θ such that $\tan(\theta) = 1/\beta_c$. Expanding equation (2.3.6) for small values of $-\alpha$ and λ one can also obtain the slope of the tangent line to the critical curve at the Gaussian point (problem 2.3.9). Doing this we obtain for this tangent line the equation

$$\lambda(\alpha) = -\frac{1}{\beta_c}\,\frac{\alpha}{3}.$$

We see that the asymptotic slope is $-1/\beta_c$, larger therefore than the slope at the Gaussian point, which is $-1/(3\beta_c)$, by a factor of 3, thus showing that the critical curve has its concavity turned mostly upwards.

In order to get the graphs of the critical curve we are compelled to solve the equation by numerical means. In fact, this can be done not only for our $O(1)$ case here, but for the $SO(\mathfrak{N})$ generalizations as well. A curious fact is that it is in fact easier to do this for $\mathfrak{N} = 2$ and the other even-$\mathfrak{N}$ cases than for $\mathfrak{N} = 1$ and the other odd-$\mathfrak{N}$ cases, because in the even-$\mathfrak{N}$ cases it turns out that the integrals can be written in terms of the error function. For the odd-$\mathfrak{N}$ cases we must use direct numerical integration in order to solve the equation, which is a technique that can be used in all cases [38]. The graphs in figures from 2.3.1 to 2.3.4 show the critical curves obtained by such numerical means in a few cases. Each graph shows also the tangent line at the Gaussian point and the asymptotic line for large values of λ.

Figure 2.3.1 shows the solution in the case $d = 3$, for fairly small values of the parameters $-\alpha$ and λ, while figure 2.3.2 shows the same solution for larger values of the parameters. Figures 2.3.3 and 2.3.4 show the corresponding data for the case $d = 4$. It should be noted that, as one can see in the two graphs with the larger values of the parameters, the asymptotic lines actually cross the critical curves. This implies that at some location for even larger values of the parameters the critical curves must have inflection points and reverse their concavities, presumably approaching their asymptotes from below rather than from above. One can see, looking directly at the data shown in the graphs, that the critical curves do in fact slowly approach the asymptotes, but it seems that the location of the inflection point is at very large values of the parameters, that so far have not been probed numerically. It is interesting to observe that this behavior seems to be characteristic of the $O(1)$ models. In the $SO(\mathfrak{N})$ models with $\mathfrak{N} > 1$ one does not see this type of crossing, and the critical curves seem to approach their asymptotes from above.

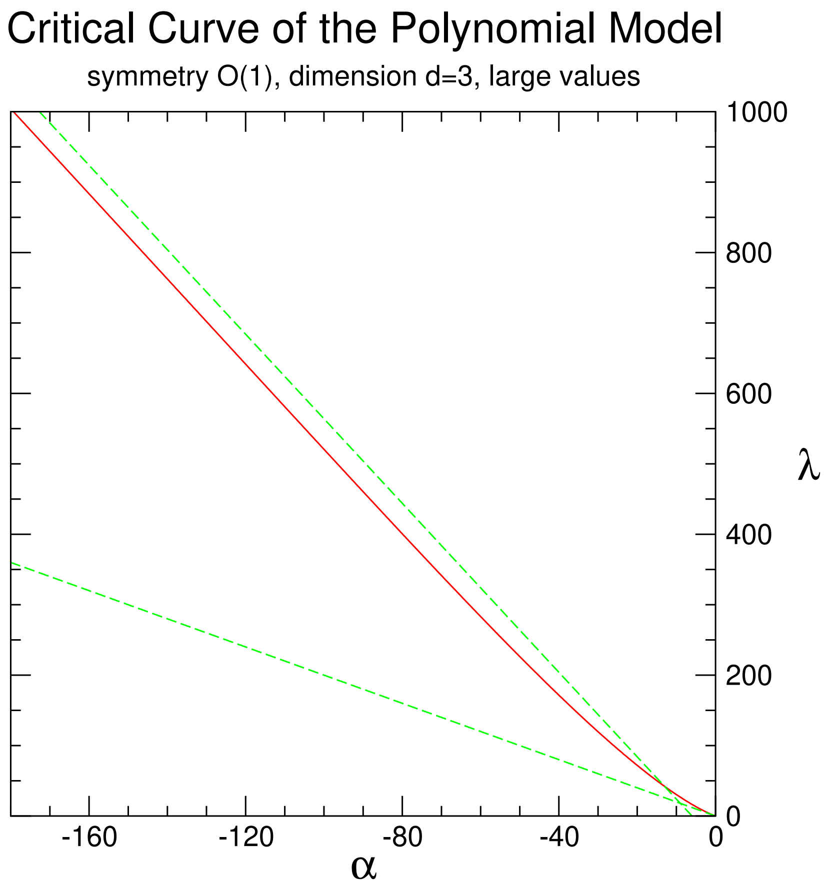

Figure 2.3.2: The mean-field critical curve of the $\lambda\varphi^4$ model with $O(1)$ symmetry, in $d = 3$, for the larger values of the parameters.

At the critical curve, where $v_R = 0$, this reduces to

$$\sigma^2_{\rm MF} = \frac{\displaystyle\int_0^\infty {\rm d}\varphi\;\varphi^2\;e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}}{\displaystyle\int_0^\infty {\rm d}\varphi\;e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}}.$$

Note that the right-hand side of this equation is exactly equal to the right-hand side of equation (2.3.6), which determines the critical line, and that therefore we have

$$\sigma^2_{\rm MF} = \frac{1}{2d} = \beta_c,$$

so that the equation of the tangent line to the mean-field critical curve at the Gaussian point can be written as

$$\lambda(\alpha) = -\frac{1}{\sigma^2_{\rm MF}}\;\frac{\alpha}{3}.$$

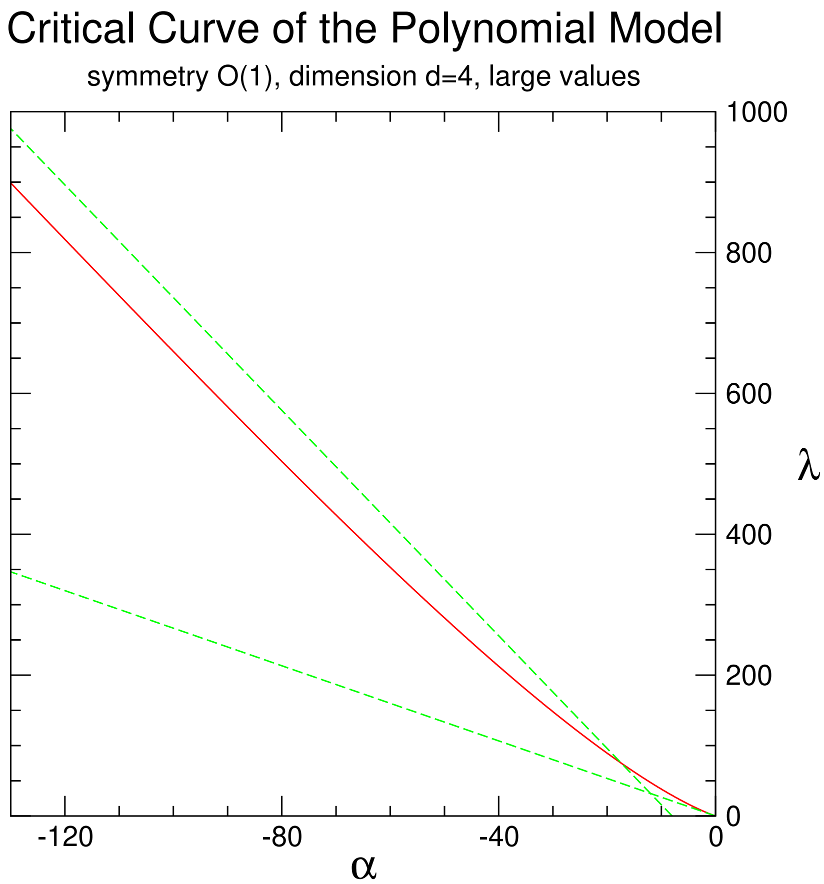

Figure 2.3.4: The mean-field critical curve of the $\lambda\varphi^4$ model with $O(1)$ symmetry, in $d = 4$, for the larger values of the parameters.

$$\mathrm{d}(v_R^2) = C_\alpha \, \mathrm{d}\alpha + C_\lambda \, \mathrm{d}\lambda,$$

where the coefficients C_α e C_λ are finite and non-vanishing expressions in the vicinity of the critical curve. The fact that we are able to write the differential of v_R^2 directly in terms of the differentials of α and λ with coefficients that are finite and not zero over the critical curve is sufficient to show that the critical exponent of v_R is, once more, $\gamma = 1/2$.

In conclusion, we have discovered that the mean-field method is related, through its generalizations, to systems on finite lattices with fixed and self-consistent boundary conditions. These systems define a second family of continuum limits for the models, which does not necessarily have to be identical to the one defined by the systems with periodical boundary conditions. Although the two families of limits have, by and large, similar properties, many of the details are not identical and some quantities of interest may, in fact, depend on the boundary conditions. For example, it may be that only part of the discrepancies shown in table 2.3.1 for the critical

2.3.4. Starting from the relation $\nu_R \approx C\,(\beta - \beta_c)^\gamma$ that defines the critical exponent γ of ν_R, write the corresponding relation for the parameter $A = 2d\,\beta\nu_R$ and differentiate the resulting equation with respect to β, thus obtaining an expression for the critical exponent,

$$\gamma = \frac{\beta - \beta_c}{A}\frac{\mathrm{d}A}{\mathrm{d}\beta} - \frac{\beta - \beta_c}{\beta}.$$

In the vicinity of the critical point we have $\beta \approx \beta_c$ and $A \approx 0$, so that we may write for γ

$$\gamma = \frac{\beta - \beta_c}{A}\frac{\mathrm{d}A}{\mathrm{d}\beta}. \tag{2.3.7}$$

In order to calculate $\mathrm{d}A/\mathrm{d}\beta$, differentiate equation (2.3.1) with respect to A, and thus obtain the result

$$\frac{\mathrm{d}A}{\mathrm{d}\beta} = \frac{2d\,\sinh^2(A)}{\sinh(A)\cosh(A) - A}.$$

Use this in the expression for γ and expand it to second order around the critical point, then use the result $\beta_c = 1/(2d)$, in order to obtain for γ the expression

$$\gamma = \frac{3}{2C^2\beta_c}(\beta - \beta_c)^{1-2\gamma}.$$

Note that, given the second-order nature of the phase transition in these models, we know that γ must be within the interval $(0, 1]$. Examine the behavior of this expression in the cases $1/2 < \gamma \leq 1$ and $0 < \gamma < 1/2$, and show by reduction to absurd that the only possible value for the critical exponent is $\gamma = 1/2$. Substitute this value in the equation above and obtain the value of the constant C, thus reproducing equation (2.3.4) which was derived in the text.

2.3.5. Use the series expansions of the hyperbolic functions $\sinh(B)$ and $\cosh(B)$, with $B = 2d\,v_R\varphi$, that appear in equation (2.3.5), in order to rewrite that equation in the form

$$\frac{1}{2d} = \frac{\displaystyle\sum_{k=0}^{\infty}\frac{(2d\,v_R)^{2k}}{(2k+1)!}\int_0^\infty \mathrm{d}\varphi\;\varphi^{2k+2}\;e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}}{\displaystyle\sum_{k=0}^{\infty}\frac{(2d\,v_R)^{2k}}{(2k)!}\int_0^\infty \mathrm{d}\varphi\;\varphi^{2k}\;e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}}.$$

Obtain the same result in another way, using in equation (2.3.8) the asymptotic expansion of $\mathbf{D}_\nu$ [25], since for $\lambda \to 0$ the arguments of the $\mathbf{D}_\nu$ functions that appear in that equation go to infinity.

2.3.10. In order to determine the mean-field value of the critical exponent of v_R in the $\lambda\varphi^4$ model, write equation (2.3.5) in the form

$$\begin{aligned} v_R \int_0^\infty & \mathrm{d}\varphi\ \cosh(B)\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]} \\ = & \int_0^\infty \mathrm{d}\varphi\ \varphi\ \sinh(B)\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]}, \end{aligned}$$

where $B = 2d\, v_R\varphi$. Then differentiate in terms of $\mathrm{d}\alpha$ and $\mathrm{d}\lambda$, keeping in mind that v_R is a function of α and λ, in order to write the differential of v_R in terms of its gradient as

$$\mathrm{d}v = \frac{C_1}{C_0}\,\mathrm{d}\alpha + \frac{C_2}{C_0}\,\mathrm{d}\lambda,$$

where the coefficients are given by

$$\begin{aligned} C_0 &= \int_0^\infty \mathrm{d}\varphi\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]} \\ &\quad \times \left\{ \left[1 - 2d\,\varphi^2\right]\cosh(B) + B\ \sinh(B) \right\}, \\ C_1 &= \frac{1}{2}\int_0^\infty \mathrm{d}\varphi\ \varphi^2\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]} \\ &\quad \times \left[v\ \cosh(B) -\ \varphi\ \sinh(B)\right], \\ C_2 &= \frac{1}{4}\int_0^\infty \mathrm{d}\varphi\ \varphi^4\ e^{-\left[(d+\alpha/2)\varphi^2+(\lambda/4)\varphi^4\right]} \\ &\quad \times \left[v\ \cosh(B) -\ \varphi\ \sinh(B)\right]. \end{aligned}$$

Next expand the hyperbolic functions in each one of these coefficients for small values of B, that is, in the vicinity of the critical curve. Observe that it is enough to expand C_1 e C_2 to first order but that C_0 must be expanded to the next non-vanishing order, because the equation of the critical curve implies that the terms of orders zero and one of its expansion cancel each other. In this way, obtain the differential of v_R^2 as

$$\mathrm{d}(v^2) = C_\alpha\ \mathrm{d}\alpha + C_\lambda\ \mathrm{d}\lambda,$$

where the coefficients, which define the gradient of v_R^2, are given by

Chapter 3

Interactions Between Particles

3.1 The Coupling Constant

Having discussed in sections 1.2 and 1.3 the behavior of the $\lambda\varphi^4$ model with respect to the one-point and two-point correlation functions, which are related respectively to the phenomena of spontaneous symmetry breaking and of the propagation of waves and particles, we will now consider the three-point and four-point functions, which are related to the phenomena of interaction between waves or between particles within the model. Our first task will be to discuss the nature of the renormalized (or physical) coupling constant λ_R, and of its dimensionfull version Λ_R, relating them to expectation values of observables of the model. In this way we will define these quantities and determine, at least in principle, the way to calculate them.

As we will see, the renormalized coupling constant is a quantity that vanishes in the Gaussian model and whose value measures how non-Gaussian the renormalized ensemble of the model under study is, thus determining its true degree of non-linearity and the existence or not, within the structure of the model, of phenomena of interaction between waves or between particles. Note that we are not talking here about the ensemble of the fundamental field, but rather about the ensemble of the physical variables associated to blocks, as was discussed in the section in reference [6], since these are the variables that are directly associated to the actual physical observables of the theory. Hence, we should expect the quantity of greater interest in this discussion to be the *dimensionfull* renormalized coupling constant Λ_R, since it is the dimensionfull quantities that scale in the correct way and thus are related to the block-variable observables, as we saw explicitly in the case of the propagator in the section in reference [6].

In order to be able to write the renormalized coupling constant in terms of observables of the model, we return to the discussion of the formalism of the generating functionals and of the effective action, which were introduced in the sections in references [39] and [45]. We saw in the section in reference [39] that the complete Green functions $g_{1,\ldots,n} = \langle\varphi_1 \ldots \varphi_n\rangle$ of the theory in the absence of external sources can be obtained by means of multiple functional differentiations with respect to j of the functional $Z[j]$ defined in the equation in reference [41], after which one makes consists of a relatively small number of terms with different structures, each one accompanied of all the possible permutations of the position indices. Observe that in any circumstances in which $\varphi_{(c)} = 0$, corresponding necessarily to $j = 0$, we obtain a much simpler relation, that can be written as

$$g_{(c)1,2,3,4} = g_{1,2,3,4} - [g_{1,2}\, g_{3,4} + g_{1,3}\, g_{2,4} + g_{1,4}\, g_{2,3}], \tag{3.1.2}$$

since for $j = 0$ we have that $g_{(c)i,j} = g_{i,j}$. Just as $g_{(c,j)i,j}$ gives us the true two-point correlations, $g_{(c,j)i,j,k,l}$ gives us the true four-point correlations of the model, that is, those which are not just superpositions of two-point correlations. While the two-point function is related to the propagation of waves and particles, the four-point function is related to the interaction between these waves and between these particles. It gives us the part of the complete four-point function which is not just a product of two-point functions. The part of the complete four-point function that can be decomposed into such a product of two-point functions corresponds to two waves or particles propagating together in a region of space-time, but that superpose linearly, passing transparently through each other, without interacting with one another. This is in fact all that happens in the theory of the free scaler field. One can show (problem 3.1.2) that in that theory, where everything is linear, the connected four-point function vanishes identically, a fact which corresponds to the lack of interactions between waves or between particles in that theory.

In order to relate this function with the renormalized coupling constant we must go back to the discussion of the concept of the effective action, which was introduced in the section in reference [39] and discussed in detail in the section in reference [45]. The renormalized coupling constant is one of the parameters that appears in the expression of the effective action, it is the parameter that relates most directly to the connected four-point function and that encodes in the most concise way the structure of interactions of the theory. As we saw, the effective action $\Gamma[\varphi_{(c)}]$ is a functional of $\varphi_{(c)}$ defined from $W[j]$ by a Legendre transformation. As we saw in the equation in reference [47], the double functional derivative of $\Gamma[\varphi_{(c)}]$ with respect to $\varphi_{(c)}$ is related to the inverse of the propagator. Starting from that equation we may write, with some changes of indices and an additional sum over the lattice, the equation

$$\sum_{3,4} g_{(c,j)1,3}\; g_{(c,j)2,4} \frac{\partial^2 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)3} \partial \varphi_{(c)4}} = g_{(c,j)1,2}.$$

If we differentiate this once more with respect to j, using then the chain rule in order to rewrite the derivatives as derivatives with respect to $\varphi_{(c)}$, as we did in the derivation of the equation in reference [47], and recalling also that the double functional derivative of $\Gamma[\varphi_{(c)}]$ is the inverse of the propagator, we obtain (problem 3.1.3)

$$\sum_{4,5,6} g_{(c,j)1,4}\; g_{(c,j)2,5}\; g_{(c,j)3,6} \frac{\partial^3 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)4} \partial \varphi_{(c)5} \partial \varphi_{(c)6}} = -g_{(c,j)1,2,3}.$$

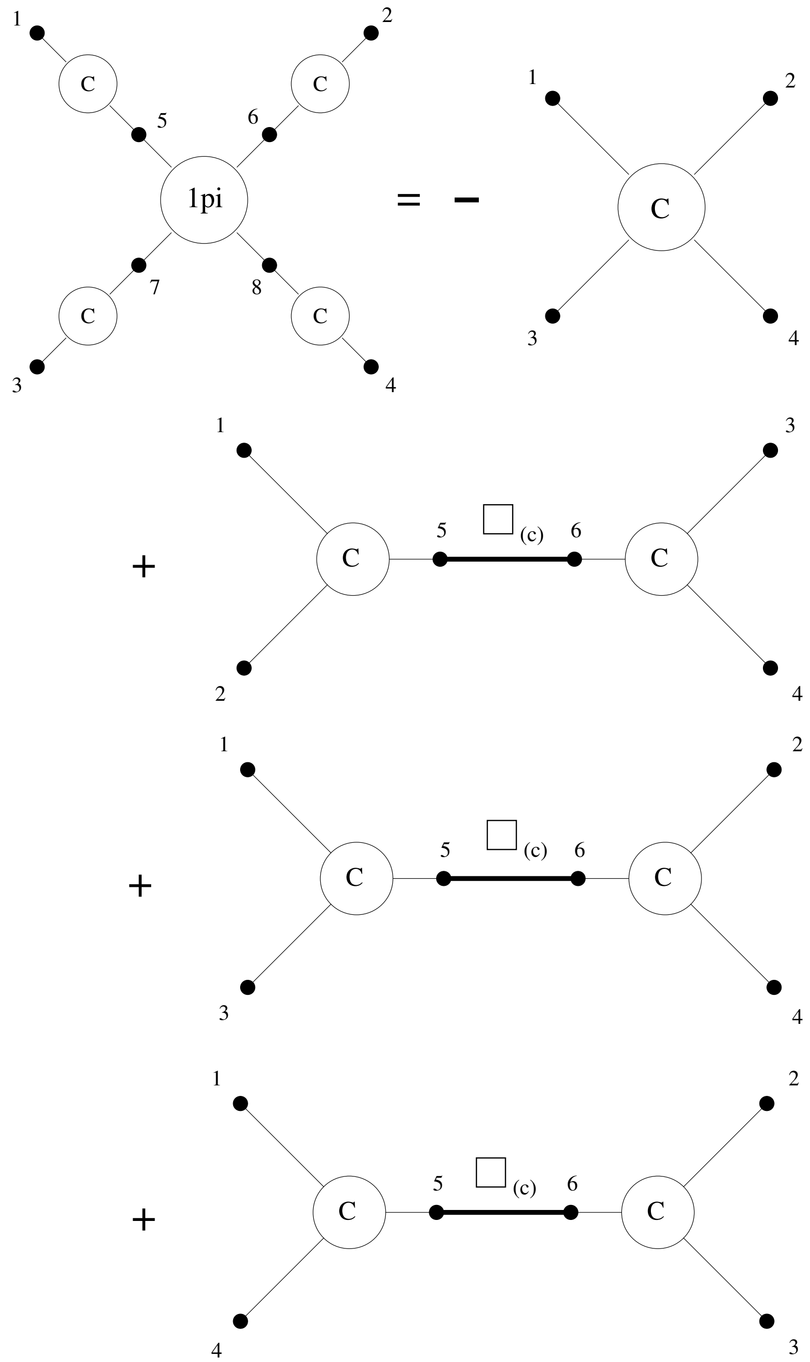

Figure 3.1.2: Diagrammatic representation of the equation for the four-point function.

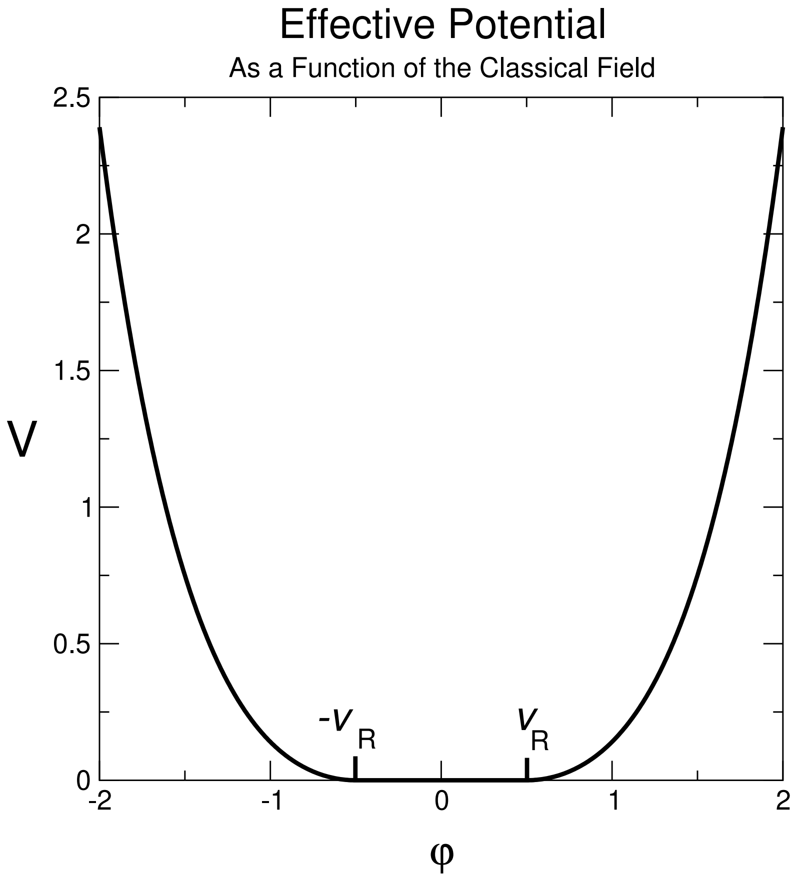

Figure 3.1.3: The effective potential as a function of the classical field $\varphi_{(c)}$.

$$\widetilde{\gamma}_{1,2,3,4} = \frac{-1}{N^{4d}} \, \frac{\widetilde{g}_{(c)1,2,3,4}}{\widetilde{g}_{(c)1} \, \widetilde{g}_{(c)2} \, \widetilde{g}_{(c)3} \, \widetilde{g}_{(c)4}}. \tag{3.1.5}$$

We have here the 1pi function written in terms of expectation values of the model in momentum space.

In order to relate this to the renormalized coupling constant, we will have to make some assumptions about the form that the effective action $\Gamma[\varphi_{(c)}]$ may have, which will be based on the symmetries of the model. Let us recall then that our polynomial model is defined by the action given in equation (1.1.1), which we reproduce here,

$$S[\varphi] = \frac{1}{2}\sum_{\ell}(\Delta_{\ell}\varphi)^2 + \frac{\alpha}{2}\sum_{s}\varphi^2(s) + \frac{\lambda}{4}\sum_{s}\varphi^4(s).$$

We will be interested primarily in analyzing the low-momentum regime of the model, because this is enough for obtaining the value of the renormalized coupling constant, since it appears as part of a local potential, which exists even for fields which are constant over the lattice, having therefore infinite wavelength and vanishing momenta. In addition to this, we will assume that the effective action has the same the effective potential as shown in figure 3.1.4. Observe that in the continuum limit we have $v_R \to 0$, since it is necessary that we approach the critical curve in the limit, where $v_R = 0$ since the phase transition is second-order. Hence, if we will end up by taking the limit, we can do the analysis either in terms of $\varphi'_{(c)}$ or in terms of $\varphi_{(c)}$. For simplicity, we will limit ourselves here to the derivation of the relation between λ_R and the observables of the model in the case in which $j = 0$, but it is not difficult to generalize the result (problem 3.1.6).

Since we will assume that the effective action, when written in terms of $\varphi'_{(c)}$, has the same symmetries that the fundamental action which defines the model has, it follows that $\Gamma[\varphi'_{(c)}]$ must be composed of terms that have the same symmetries of the terms existing in $S[\varphi]$, that is, that it must have the general form

$$\Gamma[\varphi'_{(c)}] = \frac{1}{\zeta}\left\{\frac{1}{2}\sum_{\ell}[\Delta_\ell \varphi'_{(c)}]^2 + \frac{\alpha_R}{2}\sum_{s}\varphi'^2_{(c)}(s) + \frac{\lambda_R}{4}\sum_{s}\varphi'^4_{(c)}(s) + \text{ (others) }\right\},$$

where we wrote explicitly the terms which are relevant for the analysis of the low-momentum regime of the model, ζ is the residue of the pole of the propagator and "others" indicates terms with more than four powers of the field and terms with more than two derivatives. Terms with many derivatives do not contribute significantly to the low-momentum regime and terms with more than four powers of the field do not contribute to the four-point function. Based on the numerical experience with this model, we may assume that $\zeta = 1$, which seems to be true with significant precision in all cases examined so far.

When taking the functional derivatives of $\Gamma[\varphi'_{(c)}]$, and considering that we are interested in the case $j = 0$, we should realize that it is implicit that we should put $\varphi'_{(c)} = 0$ at the end of the calculations, because this is the value of $\varphi'_{(c)}$ that corresponds to the condition $j = 0$ in this model. It is clear that the quadratic terms will vanish anyway when we take the derivatives, while the terms with powers larger than four will vanish due to the condition $\varphi'_{(c)} = 0$. Therefore, we may consider only the terms of the effective action that contain exactly four powers of the field and no derivatives, and so we are reduced to considering only the term

$$V_{(4)}[\varphi'_{(c)}] = \frac{\lambda_R}{4}\sum_{0}\varphi'^4_{(c)0},$$

which is the term of the effective potential which is relevant for zero momentum. Taking the first derivative we get

$$\frac{\partial V_{(4)}}{\partial \varphi'_{(c)1}} = \lambda_R \sum_{0} \varphi'^3_{(c)0}\delta_{0,1} = \lambda_R \varphi'^3_{(c)1}.$$

Multiplying this equation by $f_1(1)$, where, in order to simplify the notation, we are denoting the mode functions of the Fourier basis as

$$f_i(j) = e^{\imath\frac{2\pi}{N}\vec{n}_i\cdot\vec{k}_j},$$

$$\lambda_R = -\frac{1}{6N^d} \frac{\widetilde{g}_{(c)1,2,3,4}}{\widetilde{g}_{(c)1}\,\widetilde{g}_{(c)2}\,\widetilde{g}_{(c)3}\,\widetilde{g}_{(c)4}}.$$

Naturally, since we neglected the terms in $\Gamma[\varphi'_{(c)}]$ with larger powers of the momenta, this relation only makes sense for small or vanishing momenta. We take, therefore, the zero-momentum case $\vec{k}_1 = \vec{k}_2 = \vec{k}_3 = \vec{k}_4 = \vec{0}$ in order to obtain, substituting the connected functions in terms of the complete functions,

$$\lambda_R = \frac{1}{6N^d} \frac{3\langle|\widetilde{\varphi}(\vec{0})|^2\rangle^2 - \langle|\widetilde{\varphi}(\vec{0})|^4\rangle}{\langle|\widetilde{\varphi}(\vec{0})|^2\rangle^4}. \tag{3.1.7}$$

If we recall the factorization relations of the free theory for the correlation functions in momentum space, that were introduced in the section of reference [44], we immediately see that this quantity vanishes identically in the free theory.

We may also write this result in terms of the dimensionfull quantities, using the appropriate scaling relations to transform φ in ϕ and λ_R in Λ_R, thus obtaining

$$\Lambda_R = \frac{1}{6L^d} \frac{3\langle|\widetilde{\phi}(\vec{0})|^2\rangle^2 - \langle|\widetilde{\phi}(\vec{0})|^4\rangle}{\langle|\widetilde{\phi}(\vec{0})|^2\rangle^4}.$$

This is the quantity whose value determines whether or not there exists in this model the phenomenon of non-linear interaction between waves, or between particles. Naturally, this quantity is of great physical interest and we will dedicate some time to the examination of its properties.

Problems

3.1.1. Using the definition of the connected four-point correlation function

$$g_{(c,j)1,2,3,4} = \frac{\mathfrak{d}^4 W[j]}{\mathfrak{d} j_1 \mathfrak{d} j_2 \mathfrak{d} j_3 \mathfrak{d} j_4},$$

in a theory with a non-vanishing external source j, show that it is related to the complete functions of four, three and two points by the formula

$$\begin{aligned} g_{(c,j)1,2,3,4} &= g_{(j)1,2,3,4} \\ &- \left[g_{(j)2,3,4}\,\varphi_{(c)1} + g_{(j)1,3,4}\,\varphi_{(c)2} + g_{(j)1,2,4}\,\varphi_{(c)3} + g_{(j)1,2,3}\,\varphi_{(c)4}\right] \\ &- \left[g_{(j)1,2}\,g_{(j)3,4} + g_{(j)1,3}\,g_{(j)2,4} + g_{(j)1,4}\,g_{(j)2,3}\right] \\ &+ 2\big[g_{(j)1,2}\,\varphi_{(c)3}\,\varphi_{(c)4} + g_{(j)1,3}\,\varphi_{(c)2}\,\varphi_{(c)4} + g_{(j)1,4}\,\varphi_{(c)2}\,\varphi_{(c)3} \\ &\quad + g_{(j)2,3}\,\varphi_{(c)1}\,\varphi_{(c)4} + g_{(j)2,4}\,\varphi_{(c)1}\,\varphi_{(c)3} + g_{(j)3,4}\,\varphi_{(c)1}\,\varphi_{(c)2}\big] \\ &- 6\,\varphi_{(c)1}\,\varphi_{(c)2}\,\varphi_{(c)3}\,\varphi_{(c)4}. \end{aligned}$$

Next, use the fact that the second functional derivative of $\Gamma[\varphi_{(c)}]$ is the inverse of the propagator and rearrange the terms in order to obtain the final relation in the form

$$\sum_{4,5,6} g_{(c,j)1,4}\; g_{(c,j)2,5}\; g_{(c,j)3,6} \frac{\partial^3 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)4} \partial \varphi_{(c)5} \partial \varphi_{(c)6}} = -g_{(c,j)1,2,3}.$$

3.1.4. Starting from the final result of equation (3.1.3), differentiate it once again with respect to j, using the chain rule to rewrite the derivatives as derivatives with respect to $\varphi_{(c)}$, thus obtaining

$$\begin{aligned}
&\sum_{5,6,7,8} g_{(c,j)1,5}\; g_{(c,j)2,6}\; g_{(c,j)3,7}\; g_{(c,j)4,8} \frac{\partial^4 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6} \partial \varphi_{(c)7} \partial \varphi_{(c)8}} \quad + \\
&\qquad + \sum_{5,6,7} g_{(c,j)1,4,5}\; g_{(c,j)2,6}\; g_{(c,j)3,7} \frac{\partial^3 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6} \partial \varphi_{(c)7}} \quad + \\
&\qquad + \sum_{5,6,7} g_{(c,j)1,5}\; g_{(c,j)2,4,6}\; g_{(c,j)3,7} \frac{\partial^3 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6} \partial \varphi_{(c)7}} \quad + \\
&\qquad + \sum_{5,6,7} g_{(c,j)1,5}\; g_{(c,j)2,6}\; g_{(c,j)3,4,7} \frac{\partial^3 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6} \partial \varphi_{(c)7}} \quad = \quad g_{(c,j)1,2,3,4}.
\end{aligned}$$

Using again the final result of problem 3.1.3 and, once more, the fact that the second functional derivative of $\Gamma[\varphi_{(c)}]$ is the inverse of the propagator, obtain the final relation

$$\begin{aligned}
&\sum_{5,6,7,8} g_{(c,j)1,5}\; g_{(c,j)2,6}\; g_{(c,j)3,7}\; g_{(c,j)4,8} \frac{\partial^4 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6} \partial \varphi_{(c)7} \partial \varphi_{(c)8}} \\
&= -g_{(c,j)1,2,3,4} \quad + \sum_{5,6} g_{(c,j)1,2,5} \frac{\partial^2 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6}} g_{(c,j)3,4,6} \\
&\qquad + \sum_{5,6} g_{(c,j)1,3,5} \frac{\partial^2 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6}} g_{(c,j)2,4,6} \\
&\qquad + \sum_{5,6} g_{(c,j)1,4,5} \frac{\partial^2 \Gamma[\varphi_{(c)}]}{\partial \varphi_{(c)5} \partial \varphi_{(c)6}} g_{(c,j)2,3,6}.
\end{aligned}$$

3.1.5. Starting from equation (3.1.4), write it in the form

$$\sum_{5,6,7,8} g_{(c)1,5}\; g_{(c)2,6}\; g_{(c)3,7}\; g_{(c)4,8}\; \Gamma_{5,6,7,8} = -g_{(c)1,2,3,4}$$

and execute four Fourier transforms on the external variables $\vec{n}_1, \ldots, \vec{n}_4$, using the corresponding variables $\vec{k}_1, \ldots, \vec{k}_4$ in momentum space, and recalling that, for a function F of n position variables $\vec{n}_i$, for the calculation of the propagator, we obtain (problem 3.2.1) the classical result $\lambda_R = \lambda$. Naturally, this implies that $\Lambda_R = \Lambda$ to first order in ε, which means to first order in λ, which seems to indicate, in a superficial way, that we may have a non-vanishing renormalized coupling constant in the quantum theory. We will see, however, that this is an excessively superficial analysis and that the real situation is much more complex than what it indicates.

Let us recall that, in the calculation of the propagator, the calculation of order zero in ε resulted in the classical propagator, that is, in the propagator of the Gaussian ensemble that we are using to approximate the ensemble of the complete theory, while the calculation to first order in ε gave us quantum corrections to the classical result. Also, this first-order result was not a small correction of the zero-order result, but instead was qualitatively different from it. In the case of the coupling constant it is clear that the zero-order calculation results in $\lambda_R = 0$, which is the value of this quantity in the Gaussian ensemble, while the first-order calculation gives us the classical result. This situation is to be expected, since we are now calculating a quantity which is, by definition, at least proportional to the expansion parameter λ, and that vanishes when the expansion parameter vanishes, unlike what was the case for α_R. Unlike what happened in the case of α_R, we are discussing here a quantity that does not exist at all in the Gaussian ensemble of the free theory. Just like what happened in the case of the propagator, it is possible that passing to the next order, which takes us away from the classical results, will make a qualitative difference.

These results that we refer to as "classical" correspond, in the traditional language, to Feynman diagrams with zero loops, that is, to the "tree" approximation. These results do not include the effects of the quantum fluctuations of the fields on the observables. Hence, the zero-loop approximations do not include the quantum effects contained in the theory, but only the effects of the classical dynamics of the fields. In order to include the effect of the quantum fluctuations it is necessary to do the calculations up to the lowest order of ε which includes diagrams with one loop. In the case of m_R this meant doing calculations up to the first order in ε, but in order to explore the effects of the quantum fluctuations of the fields on the renormalized coupling constant it is necessary to calculate λ_R to second order in ε, thus including diagrams with up to one loop. These calculations to order ε^2 are considerably longer and more complex than those to the first order, and involve quantities with strong divergences that behave as either N^d or as N^{2d}, all of which cancel out completely from the final results. Doing the calculation in the symmetrical phase, with vanishing momenta on the four external legs, we obtain (problem 3.2.2)

$$\lambda_R = \frac{\lambda\left[1 - 4\dfrac{\alpha_R - \alpha_0}{\alpha_0} - 9\lambda s_2(\alpha_0)\right]}{\left[1 - \dfrac{\alpha_R - \alpha_0}{\alpha_0}\right]^4},$$

where the dimensionless sum $s_2(\alpha_0)$ that appears here is given, in terms of the dimensionless free propagator $g_0(\vec{n}, \vec{n}')$, by

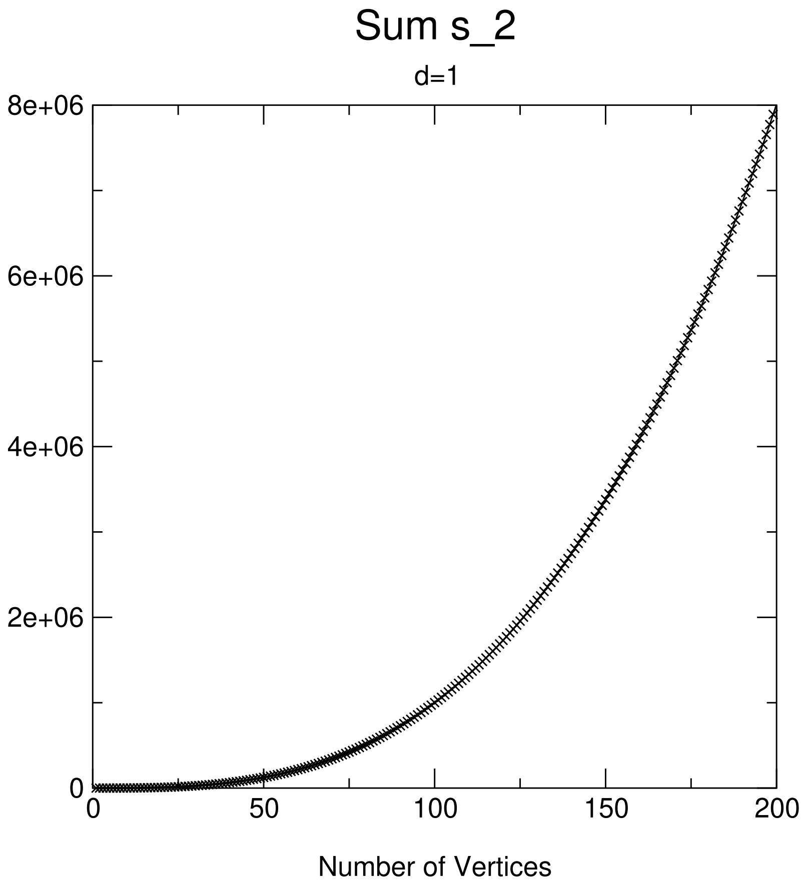

Figure 3.2.1: Behavior of the sum s_2 with N in the case $d = 1$.

Of course this type of limit constitutes a small subset of all possible alternatives, in which we approach, in the limit, some other arbitrary point of the critical curve, rather than the Gaussian point. About these other possibilities our perturbative approximation has nothing to say, but note that, is we assume that there is at least one such limit for each one of these points, in which Λ_R is finite, it follows immediately that it is necessary that $\lambda_R \rightarrow 0$ over the whole critical curve, when we make $N \rightarrow \infty$. This means that it is highly likely that the ensemble of the renormalized theory becomes Gaussian over the critical curve in the continuum limit. Since λ_R is a dimensionless quantity that measures, just like α_R, a moment of the distribution of the renormalized model, it is very reasonable to think that both should have the same particular type of behavior in the locus of the parameter plane of the model where the critical transition takes place. In other words, it is reasonable to think that λ_R should always go to zero over the critical curve in the continuum limit, as part of the critical behavior of the model.

We may also try to extract some information of the result in equation (3.2.2) in the case $d \geq 5$. In this case our result seems to indicate that we will have a finite and non-vanishing λ_R as a function of α and λ, since the sum s_2 is finite and

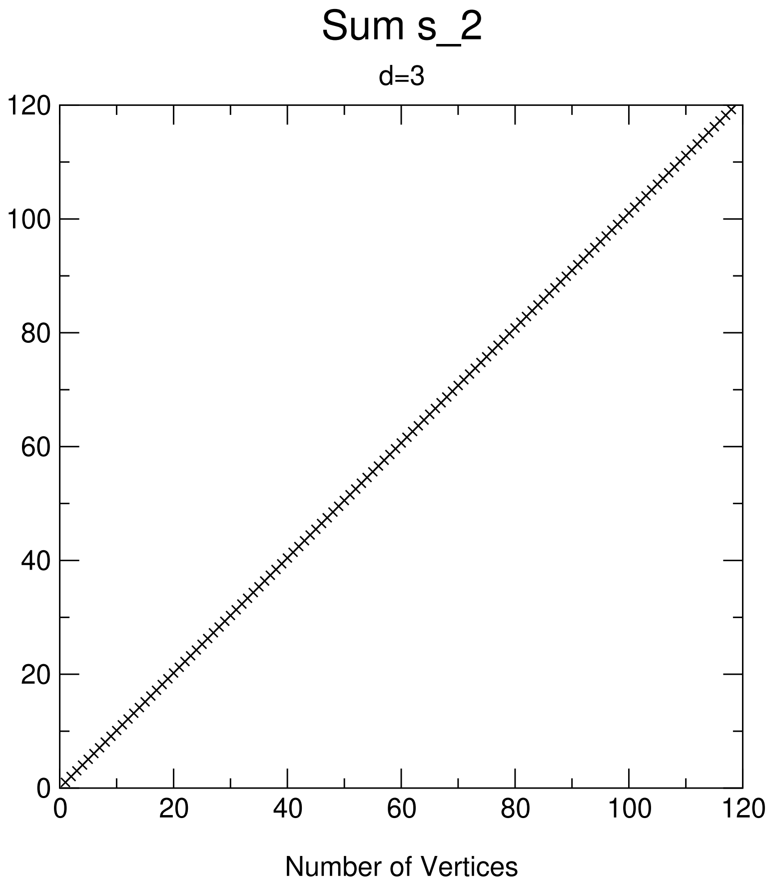

Figure 3.2.3: Behavior of the sum s_2 with N in the case $d = 3$.

lattices (problem 3.2.6). The results obtained in this way, for $m_R L = 1$, are shown in the graphs found in figures from 3.2.1 to 3.2.5. Simple but good-quality curve fittings, with only a single parameter in the cases $d = 1$, $d = 2$ and $d = 3$, and with three parameters in the cases $d = 4$ and $d = 5$, give the approximate results

$$\begin{aligned}
s_2(d = 1) &\approx 1.0N^3, \\
s_2(d = 2) &\approx 1.0N^2, \\
s_2(d = 3) &\approx 1.0N, \\
s_2(d = 4) &\approx 1.0 + 0.013 \ln(0.93N), \\
s_2(d = 5) &\approx 0.19 + 1.0\frac{1}{N} - 0.014\frac{1}{N^2}.
\end{aligned}$$

One may also consider calculating the expectation value of the field and of the propagator up to order ε^2, which corresponds to the inclusion of diagrams with up to two loops. In some cases one may still be able to extract from these calculations some useful information such as, for example, for the determination of the critical curve (problem 3.2.7) and, in some cases, for the determination of the renormalized mass (problem 3.2.8). However, in general the results of these calculations include

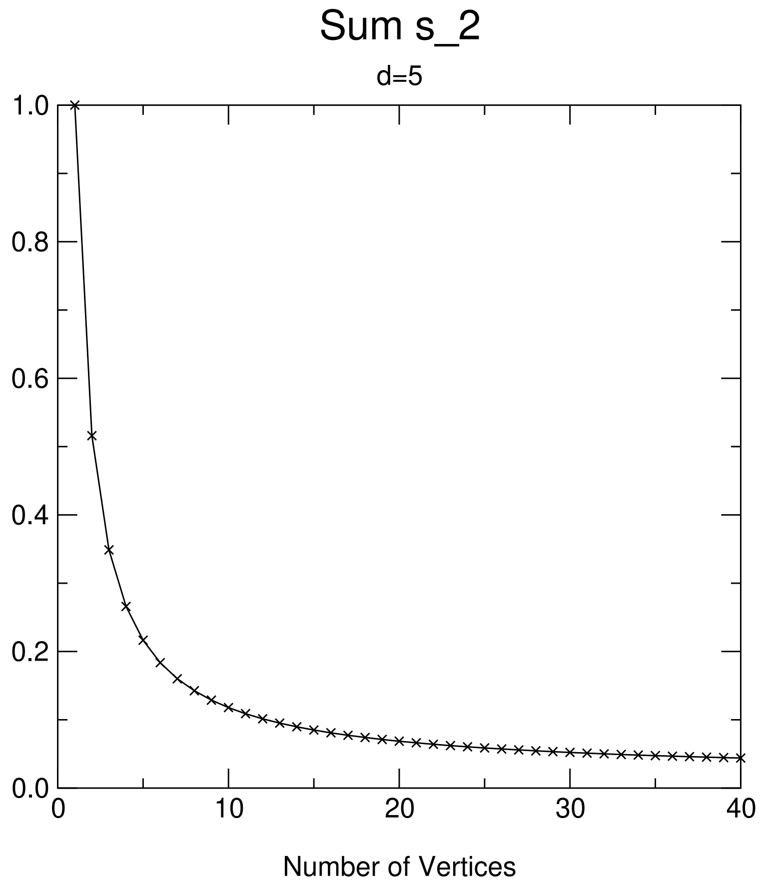

Figure 3.2.5: Behavior of the sum s_2 with N in the case $d = 5$.

a perturbative approximation as successful as in the cases of the renormalized mass and of the expectation value of the field. It allows us to go so far as to formulate the *conjecture* that, in any dimension $d \geq 3$, in any limit where $N \to \infty$, $\alpha(N) \to \alpha_{(c)}$ and $\lambda(N) \to \lambda_{(c)}$, where $(\alpha_{(c)}, \lambda_{(c)})$ is a point over the critical curve, we have $\lambda_R \to 0$. However, it does not allow us to make concrete predictions about the behavior of the model away from the Gaussian point, as we were able to do for the renormalized mass and the expectation value of the field. It seems, therefore, that the discussion of the perturbative approximation should be separated in two parts, the first one being relative to the calculation of the quantities involving, at most, the second moment of the distribution of the model, as is the case for v_R and α_R, while the second one relates to the quantities involving the higher-order moments, such as λ_R.

We see that the reason why the approximation does not work so well for the coupling constant is the fact that it is not possible to impose, in this case, the equivalent of the condition $\alpha_0 = \alpha_R$ used in the case of the renormalized mass, which transformed the first-order perturbative approximation into a self-consistent Gaussian approximation. This is due, of course, to the fact that the Gaussian model, the only one we know how to solve exactly, has no moments of order greater than two, re-expand the expressions that appear when one tries to do this, possibly neglecting higher-order terms in order to keep consistent powers of the expansion parameters. All this considerably complicates the whole argument and makes it more difficult to understand what is really going on when one does all this. Let us try to exemplify this in the context of the calculations that we made here, at first in a very simple and possibly incomplete way. We have calculated one-loop approximations for α_R and λ_R, obtaining expressions of the form

$$\alpha_R = f_\alpha(\alpha, \lambda) \quad \text{and} \quad \lambda_R = f_\lambda(\alpha, \lambda).$$

We saw that, while the one-loop propagator is entirely finite, the renormalized mas parameter being given by

$$\alpha_R = \alpha + 3\sigma_0^2\lambda,$$

the one-loop coupling constant contains the divergent sum s_2, being given by

$$\lambda_R = \lambda[1 - 8s_2\lambda].$$

We proceed then to a change of variables, introducing a new parameter λ_0 in place of λ, defined by the relation

$$\lambda = \lambda_0 + 9s_2\lambda_0^2.$$

Note that this mixes powers of λ and corresponds, therefore, to a reorganization of the perturbative expansion. Substituting this expression for λ in the result for λ_R we verify that the divergent terms of order λ_0^2 cancel out, so that we obtain

$$\lambda_R = \lambda_0 - 162s_2^2\lambda_0^3 - 729s_2^3\lambda_0^4.$$

We now argue that we can neglect in this equation the terms of orders λ_0^3 and λ_0^4, not because they are small, since they are clearly divergent in the limit, but under the allegation that they will cancel out with the remaining higher-order terms that have not yet been explicitly included in this analysis. This is the first condition involved in the criterion of perturbative renormalizability in a weak sense, term by term in the perturbative expansion, without preoccupation with its convergence. What we are requiring here is that the divergent terms cancel out, not in the original series, but after its reorganization by the change of variables from λ to λ_0. Under these conditions we have

$$\lambda_R = \lambda_0,$$

which shows that our change of variables is in fact a change from an expansion in terms of the basic parameter λ to another expansion in terms of the renormalized parameter λ_R. If we now write λ in terms of λ_R,

$$\lambda = \lambda_R + 9s_2\lambda_R^2,$$

Proceeding with the substitution of λ by $\lambda_0 = \lambda_R = \lambda_R(\vec{k} = \vec{0})$ we obtain

$$\lambda_R(\vec{k}) = \lambda_0 \left\{ 1 + 3\lambda_0 \left[s_2(\vec{0}, \alpha_0) - s_2(\vec{k}, \alpha_0) \right] \right\}. \tag{3.2.5}$$

The difference of the two sums can be evaluated for small values of the momentum $\vec{k}$ with the help of approximations by integrals and, doing this in $d = 4$ and for large values of N (problem 3.2.10), we obtain a finite result,

$$\lambda_R(\vec{k}) = \lambda_R \left[1 + 24\pi^2 \lambda_R \frac{k^2}{L^6 m_R^6} \right].$$

We say that the theory is perturbatively renormalizable if it is possible to do this in each order of perturbation theory, and hence to obtain predictions with arbitrarily high precision for $\mathcal{O}$, given values of α_R and λ_R. Note that, if we imagine that the theory is in fact trivial, then we see that this result is not wrong, but that is is simply rather irrelevant, because in this case the only possible value for λ_R is zero and the relation simply shows that $\lambda_R(\vec{k}) = 0$ for any $\vec{k}$. We can see now that there is in fact a rather subtle problem behind all this. When we do this kind of manipulation we are giving up obtaining from the theory the relations between (α_R, λ_R) and (α, λ) and, instead of that, *we implicitly assume that certain values of* α_R *and* λ_R *are possible* in the context of the model defined in a non-perturbative way by means of the lattice. This seems to be a very reasonable thing to do in a model which is defined with two free parameters, and we certainly know which values are or are not possible for α and λ. However, we do not have now any information about *which values are in fact possible* for the renormalized parameter λ_R, according to the non-perturbative definition of the model. Therefore, we do now know which values we may in fact use for λ_R in this perturbative renormalization scheme.

It is implicitly assumed, in the traditional perturbative renormalization scheme, that the possible values for (α_R, λ_R) are the same which are possible for (α, λ). However, in general it is possible that this is not true, and that there are restrictions for the images of the relations $\alpha_R(\alpha, \lambda)$ and $\lambda_R(\alpha, \lambda)$ determined by the non-perturbative definition of the models. One restriction that we already know to exist in this model is that $\alpha_R \geq 0$, while the parameter α can be either positive or negative on finite lattices, and must become negative in the continuum limit, as we saw in section 1.3. Another fact, which is even more important than this one, is that there certainly are important restrictions for λ_R in a model that ends up being trivial, in which the only possible value for λ_R in the $N \to \infty$ limit is zero. We can always determine beforehand which values are possible for (α, λ), but we cannot do the same for (α_R, λ_R). Triviality implies that the usual implicit hypothesis, that the possible values for (α_R, λ_R) are the same which are possible for (α, λ), is false. To continue with the usual perturbative renormalization scheme under these conditions can only produce fictitious results, without any physical or mathematical relevance.

The conclusion is that a model satisfying the criterion of perturbative renormalizability is not sufficient to guarantee the usefulness of its perturbative expansion, renormalized in the usual way. It is also necessary to determine the values which are should be calculated to second order, while it is enough to calculate the denominator to first order.

3.2.3. ($\star\star\star$) Repeat the calculation of λ_R to second order proposed in problem 3.2.2, this time in the broken-symmetrical phase[2].

3.2.4. Evaluate the asymptotic behavior of the sum s_2 given in equation (3.2.1), for large values of N, approximating it by integrals over the momenta, as we did before in the case of the quantity σ_0^2 related to the propagator, for each relevant value of d. Whenever it becomes necessary to use a minimum but non-vanishing value of the modulus of the momentum as a lower integration limit, use m_R as that value.

3.2.5. Use the result in equation (3.2.4) for the dimensionfull renormalized coupling constant Λ_R in $d = 3$, as well as the one-loop result for the renormalized mass m_R obtained in section 1.3, in order to exhibit explicitly flows $[\alpha(N), \lambda(N)]$ that approach the Gaussian point in the continuum limit and for which both m_R and Λ_R have finite and non-vanishing limits. Assume, if necessary, that Λ_R is small compared to m_R. The solutions should tend asymptotically to the line tangent to the critical curve at the Gaussian point, and both $\alpha(N)$ and $\lambda(N)$ should go to zero as $1/N$. Hint: try

$$\alpha = \frac{A}{N} + \frac{B}{N^2} \quad \text{and} \quad \lambda = \frac{C}{N}.$$

3.2.6. Write programs to calculate numerically the sum s_2 given in equation (3.2.1) in dimensions from $d = 1$ to $d = 5$ and confirm the asymptotic results obtained in problem 3.2.4. These sums should be calculated with the same numerical techniques that were used for the calculation of the sums that appear in the quantity σ_0^2, which were calculated in the section in reference [49]. In fact, it suffices to make small changes in the programs written for that case in order to produce the programs needed in this case.

3.2.7. ($\star\star$) Calculate v_R to order ε^2 and thus obtain the two-loop evaluation of the equation of the critical curve. Start by calculating v_R in the broken-symmetrical phase and obtain the result

$$\begin{aligned} 0 \;=\;& \left(\lambda v^2 + \alpha + 3\lambda\sigma_0^2\right)\left(3\lambda v^2 + \alpha - 2\alpha_0 + 3\lambda\sigma_0^2\right) \\ &+\lambda\left[9\lambda v^2 + 3(\alpha - \alpha_0) + 9\lambda\sigma_0^2\right]\alpha_0 \sum_{\vec{n}} g_0^2(\vec{0}, \vec{n}) + 6\lambda^2\alpha_0 \sum_{\vec{n}} g_0^3(\vec{0}, \vec{n}). \end{aligned}$$

Next evaluate the asymptotic behavior of the new sum that appears,

$$s_3(\alpha_0) = \sum_{\vec{v}} g_0^3(\vec{0}, \vec{n}) = \frac{1}{N^{2d}} \sum_{\vec{k}_1, \vec{k}_2} \frac{1}{[\rho^2(\vec{k}_1) + \alpha_0][\rho^2(\vec{k}_2) + \alpha_0][\rho^2(\vec{k}_1 + \vec{k}_2) + \alpha_0]}.$$

[2]Note: the answer to this problem is currently unknown.

3.2.9. ($\star$) Calculate $\lambda_R(\vec{k})$ for equal non-vanishing momenta $\vec{k}$ on all the external legs, entering in two of them and going out in the other two, using for this purpose the expression for this quantity that results from problem 3.1.7, to second order, with the choice $\alpha_0 = \alpha_R$, in the symmetrical phase, obtaining the result quoted in the text,

$$\lambda_R(\vec{k}) = \lambda \left\{ 1 - 3\lambda \left[2s_2(\vec{0}, \alpha_0) + s_2(\vec{k}, \alpha_0) \right] \right\},$$

where the sum s_2 is given by

$$s_2(\vec{k}, \alpha_0) = \frac{1}{N^d} \sum_{\vec{k}_1} \frac{1}{[\rho^2(\vec{k}_1 + \vec{k}) + \alpha_0][\rho^2(\vec{k}_1 - \vec{k}) + \alpha_0]}.$$

Note that, since $\lambda_R(\vec{k})$ itself is already a first-order quantity in λ, in order to keep consistent orders of the expansion parameter the numerator of the equation that defines $\lambda_R(\vec{k})$ should be calculated to second order, while it is enough to calculate the denominator to first order.

3.2.10. Evaluate the asymptotic behavior of the difference of sums given in equation (3.2.5), for large values of N, approximating the sums by integrals over the momenta, as we did before in problem 3.2.4. Whenever it becomes necessary to use a minimum but non-vanishing value of the modulus of the momentum as a lower integration limit, use m_R as that value.

Bibliography

[1] **TODO**: fix all entries; reorganize the references to the first book; include HTML links for the web version.

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[16] *Ibid.*[3], formula 3.621, equations 3 and 4, page 369.

[17] *Ibid.*[3], formula 8.250, equation 1, page 930.

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